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A parametric superdirective beamformer with uniform linear microphone arrays

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Abstract

Superdirective beamforming has attracted much interest in acoustic, speech and audio processing since it has the potential to achieve the maximum directivity factor (DF) for noise, interference, and reverberation suppression. However, the superdirective beamformer is sensitive to sensors' noise and mismatch between sensors, which considerably restricts its use in practical systems. Therefore, how to achieve a relatively large DF with a reasonable white noise gain (WNG) is becoming an important issue in superdirective beamforming. This paper studies this problem based on the use of a parametric gain as the cost function, which combines the DF and the WNG in one single formula. By maximizing this gain, we derive a parametric superdirective beamformer. Through properly choosing the parameter order within a small range, this beamformer can achieve a good compromise between a high value of the DF and a low value of the WNG.

Keywords: Microphone arrays, uniform linear arrays, superdirective beamforming, robust beamforming, white noise gain, directivity factor.

1 Introduction

Beamforming methods are widely used in sensor array systems to recover a desired signal from noisy observations [1–5]. A considerable amount of attention has been paid to this area of research and many beamforming algorithms have been developed over the last few decades [6–11]. Among those, the superdirective beamformer is very attractive in applications with small-size microphone arrays; it is obtained from maximization of the directivity factor (DF), i.e., gain of the signal-to-noise ratio (SNR) in a spherically isotropic (diffuse) noise field [7, 12, 13]. The superdirective beamformer is a fixed beamformer since the noise pseudo-coherence matrix is time invariant and data independent [7, 10]. In fact, when the microphone array size is very small, the superdirective beamformer corresponds to the hypercardioid, which is also derived by maximizing the DF [10]. While it is efficient in dealing with diffuse noise, the superdirective beamformer is very sensitive to sensors' noise and array imperfections, which considerably restricts its use in practical systems [14, 15]. Therefore, how to achieve a relatively high DF with a reasonable value of white noise gain (WNG) is becoming an important issue regarding the design of superdirective beamforming.

A significant amount of research has been devoted to circumventing this fundamental issue. One of the most used solutions is the regularized superdirective beamformer [1, 11, 14], the performance of which is controlled by a regularization parameter [1]. In practice, however, it is not easy to find the optimal value of this parameter since it is frequency dependent and varies from zero to infinity. There are many other popular approaches to this issue, such as the probability based beamformers, nonlinear optimization based beamformers, combined beamformers [16], and subspace beamformers [17]. In [16], the authors combined the regularized superdirective beamformer together with the delay-and-sum (DS) beamformer leading to a robust regularized superdirective beamformer, which can make a tradeoff between DF and WNG. In [17], the problem of superdirective beamforming was cast to a framework with matrix joint diagonalization and a subspace superdirective beamformer was derived, which can achieve a good compromise between a high DF and white noise amplification.

This paper is devoted to finding other alternatives to the robust superdirective beamformer. A novel parametric superdirective beamformer is derived from the maximization of a parametric SNR gain (with $1/p$ order noise pseudo-coherence matrix, where $p \in [1, \infty]$ is the parameter). With the proposed beamformer, a good compromise between a high value of the DF and a low value of the WNG can be obtained by properly choosing the parameter order within a small range, which makes it easier to use than the conventional regularized superdirective beamformer.

2 Signal Model, Performance Measures, and Conventional Beamformers

We consider the signal model in which a farfield source signal (plane wave) propagates from the direction (azimuth angle) θ in an anechoic acoustic environment at the speed of sound, i.e., $c = 340$ m/s, and impinges on a uniform linear sensor array consisting of M omnidirectional

microphones. In this context, the steering vector (of length M) is given by

$$\mathbf{d}\left(\omega, \theta\right) = \left[\begin{array}{cccc} 1 & e^{-\jmath\omega\tau_0\cos\theta} & \cdots & e^{-\jmath(M-1)\omega\tau_0\cos\theta} \end{array}\right]^T, \tag{1}$$

where the superscript ^{T} is the transpose operator, $\jmath = \sqrt{-1}$ is the imaginary unit, $\omega = 2\pi f$ is the angular frequency, $f > 0$ is the temporal frequency, and $\tau_0 = \delta/c$ is the delay between two successive sensors at the angle $\theta = 0$, with δ being the interelement spacing.

Our main interest in this paper is in superdirective beamforming [1], [7]. Therefore, it is assumed that δ is small and the desired source propagates from the endfire, i.e., $\theta = 0$ [10]. As a result, the observation signal vector is [6]

$$\begin{aligned} \mathbf{y}\left(\omega\right) &= \left[\begin{array}{cccc} Y_1\left(\omega\right) & Y_2\left(\omega\right) & \cdots & Y_M\left(\omega\right) \end{array}\right]^T \\ &= \mathbf{x}\left(\omega\right) + \mathbf{v}\left(\omega\right) \\ &= \mathbf{d}\left(\omega\right)X\left(\omega\right) + \mathbf{v}\left(\omega\right), \end{aligned} \tag{2}$$

where

$$Y_m\left(\omega\right) = e^{-\jmath(m-1)\omega\tau_0}X\left(\omega\right) + V_m\left(\omega\right) \tag{3}$$

is the signal picked up by the mth microphone, $X\left(\omega\right)$ is the desired signal, $V_m\left(\omega\right)$ is the additive noise at the mth microphone, $\mathbf{x}\left(\omega\right) = \mathbf{d}\left(\omega\right)X\left(\omega\right)$ with $\mathbf{d}\left(\omega\right) = \mathbf{d}\left(\omega, 0\right)$, and $\mathbf{v}\left(\omega\right)$ is defined similarly to $\mathbf{y}\left(\omega\right)$.

Generally, beamforming is performed by applying a complex-valued weight to the output of each sensor [6], i.e.,

$$\begin{aligned} Z\left(\omega\right) &= \sum_{m=1}^{M} H_m^*\left(\omega\right)Y_m\left(\omega\right) \\ &= \mathbf{h}^H\left(\omega\right)\mathbf{y}\left(\omega\right) \\ &= \mathbf{h}^H\left(\omega\right)\mathbf{d}\left(\omega\right)X\left(\omega\right) + \mathbf{h}^H\left(\omega\right)\mathbf{v}\left(\omega\right), \end{aligned} \tag{4}$$

where $Z\left(\omega\right)$ is the estimate of the desired signal, $X\left(\omega\right)$, the superscript * is the complex conjugate, $\mathbf{h}\left(\omega\right)$ is a linear filter of length M, and the superscript ^{H} is the conjugate-transpose operator. In our context, the distortionless constraint is desired, i.e.,

$$\mathbf{h}^H\left(\omega\right)\mathbf{d}\left(\omega\right) = 1. \tag{5}$$

In order to derive and/or evaluate different kind of optimal fixed beamformers, three fundamental performance measures are widely used. They are:

- beampattern, which is defined as

$$\begin{aligned} \mathcal{B}\left[\mathbf{h}\left(\omega\right), \theta\right] &= \mathbf{d}^H\left(\omega, \theta\right)\mathbf{h}\left(\omega\right) \\ &= \sum_{m=1}^{M} H_m\left(\omega\right)e^{\jmath(m-1)\omega\tau_0\cos\theta}, \end{aligned} \tag{6}$$

- white noise gain (WNG), which is written as

$$\mathcal{W}\left[\mathbf{h}\left(\omega\right)\right] = \frac{\left|\mathbf{h}^{H}\left(\omega\right)\mathbf{d}\left(\omega\right)\right|^{2}}{\mathbf{h}^{H}\left(\omega\right)\mathbf{h}\left(\omega\right)}, \tag{7}$$

- and directivity factor (DF), which is in the following form

$$\mathcal{D}\left[\mathbf{h}\left(\omega\right)\right] = \frac{\left|\mathbf{h}^{H}\left(\omega\right)\mathbf{d}\left(\omega\right)\right|^{2}}{\mathbf{h}^{H}\left(\omega\right)\mathbf{\Gamma}\left(\omega\right)\mathbf{h}\left(\omega\right)}, \tag{8}$$

where the elements of the $M \times M$ matrix $\mathbf{\Gamma}\left(\omega\right)$, which is the pseudo-coherence matrix corresponding to the spherically isotropic (diffuse) noise, are

$$\left[\mathbf{\Gamma}\left(\omega\right)\right]_{ij} = \frac{\sin\left[\omega(j-i)\tau_{0}\right]}{\omega(j-i)\tau_{0}} = \operatorname{sinc}\left[\omega(j-i)\tau_{0}\right]. \tag{9}$$

The beampattern measures the array response to a plane wave from the θ direction. The WNG tells how a beamformer is robust to different kinds of imperfections in the array and sensors. And the DF measures the directional gain of the beamformer.

We give below the most used and studied fixed beamformers with microphone arrays. They are:

- delay-and-sum (DS) [6]:

$$\mathbf{h}_{\mathrm{DS}}\left(\omega\right) = \frac{\mathbf{d}\left(\omega\right)}{M}, \tag{10}$$

- superdirective [7]:

$$\mathbf{h}_{\mathrm{S}}\left(\omega\right) = \frac{\mathbf{\Gamma}^{-1}\left(\omega\right)\mathbf{d}\left(\omega\right)}{\mathbf{d}^{H}\left(\omega\right)\mathbf{\Gamma}^{-1}\left(\omega\right)\mathbf{d}\left(\omega\right)}, \tag{11}$$

- and robust (or regularized) superdirective [1], [7]:

$$\mathbf{h}_{\mathrm{R},\epsilon}\left(\omega\right) = \frac{\left[\mathbf{\Gamma}\left(\omega\right)+\epsilon\mathbf{I}_{M}\right]^{-1}\mathbf{d}\left(\omega\right)}{\mathbf{d}^{H}\left(\omega\right)\left[\mathbf{\Gamma}\left(\omega\right)+\epsilon\mathbf{I}_{M}\right]^{-1}\mathbf{d}\left(\omega\right)}, \tag{12}$$

where $\epsilon \geq 0$ is a regularization parameter and $\mathbf{I}_{M}$ is the $M \times M$ identity matrix. The parameter ϵ attempts to make a compromise between a large value of DF and white noise amplification. A small value of ϵ leads to a large value of DF but a small WNG, while a large value of ϵ gives a large WNG but a low value of DF. Two interesting cases of (12) are $\mathbf{h}_{\mathrm{R},0}\left(\omega\right) = \mathbf{h}_{\mathrm{S}}\left(\omega\right)$ and $\mathbf{h}_{\mathrm{R},\infty}\left(\omega\right) = \mathbf{h}_{\mathrm{DS}}\left(\omega\right)$.

The DS beamformer is equivalent to the beamformer that maximizes WNG, and therefore with this beamformer we have $\mathcal{W}\left[\mathbf{h}_{\mathrm{DS}}\left(\omega\right)\right] = M$. The superdirective beamformer maximizes DF; in this case, $\mathcal{D}\left[\mathbf{h}_{\mathrm{S}}\left(\omega\right)\right] = \mathbf{d}^{H}\left(\omega\right)\mathbf{\Gamma}^{-1}\left(\omega\right)\mathbf{d}\left(\omega\right)$ and its value is close to M^{2} for a small value of δ [3]. It is well known that this beamformer is sensitive to sensors' noise and array imperfections. This is why $\mathbf{h}_{\mathrm{R},\epsilon}\left(\omega\right)$ was derived by adding a constraint on the WNG.

The objective of this paper is to find other alternatives to the regularized superdirective beamformer, $\mathbf{h}_{\mathrm{R},\epsilon}\left(\omega\right)$, and better ways to compromise between WNG and DF.

3 Parametric Superdirective Beamformer

Using eigenvalue decomposition [18], the pseudo-coherence matrix of the diffuse noise can be decomposed as

$$\mathbf{\Gamma}\left(\omega\right) = \mathbf{U}\left(\omega\right)\mathbf{\Lambda}\left(\omega\right)\mathbf{U}^T\left(\omega\right), \tag{13}$$

where $\mathbf{U}\left(\omega\right)$ is an orthogonal matrix, i.e., $\mathbf{U}^T\left(\omega\right)\mathbf{U}\left(\omega\right) = \mathbf{U}\left(\omega\right)\mathbf{U}^T\left(\omega\right) = \mathbf{I}_M$, and $\mathbf{\Lambda}$ is a diagonal matrix whose main elements are strictly positive as $\mathbf{\Gamma}\left(\omega\right)$ is positive definite. From this decomposition, we define the $1/p$ ($p \in [1,\infty]$) order noise pseudo-coherence matrix as:

$$\mathbf{\Gamma}^{\frac{1}{p}}\left(\omega\right) = \mathbf{U}\left(\omega\right)\mathbf{\Lambda}^{\frac{1}{p}}\left(\omega\right)\mathbf{U}^T\left(\omega\right). \tag{14}$$

Now, we introduce the parametric SNR gain, which is as follows:

$$\mathcal{G}_p\left[\mathbf{h}\left(\omega\right)\right] = \frac{\left|\mathbf{h}^H\left(\omega\right)\mathbf{d}\left(\omega\right)\right|^2}{\mathbf{h}^H\left(\omega\right)\mathbf{\Gamma}^{\frac{1}{p}}\left(\omega\right)\mathbf{h}\left(\omega\right)}. \tag{15}$$

It can be noticed that $\mathcal{G}_\infty\left[\mathbf{h}\left(\omega\right)\right] = \mathcal{W}\left[\mathbf{h}\left(\omega\right)\right]$ and $\mathcal{G}_1\left[\mathbf{h}\left(\omega\right)\right] = \mathcal{D}\left[\mathbf{h}\left(\omega\right)\right]$. In this paper, we propose a parametric superdirective beamformer, which is obtained by maximizing $\mathcal{G}_p\left[\mathbf{h}\left(\omega\right)\right]$ with a distortionless constraint, i.e.,

$$\min_{\mathbf{h}(\omega)} \mathbf{h}^H\left(\omega\right)\mathbf{\Gamma}^{\frac{1}{p}}\left(\omega\right)\mathbf{h}\left(\omega\right) \quad \text{subject to } \mathbf{h}^H\left(\omega\right)\mathbf{d}\left(\omega\right) = 1. \tag{16}$$

The solution of (16) is the parametric superdirective beamformer:

$$\mathbf{h}_{\mathrm{P},p}\left(\omega\right) = \frac{\mathbf{\Gamma}^{-\frac{1}{p}}\left(\omega\right)\mathbf{d}\left(\omega\right)}{\mathbf{d}^H\left(\omega\right)\mathbf{\Gamma}^{-\frac{1}{p}}\left(\omega\right)\mathbf{d}\left(\omega\right)}, \tag{17}$$

with $1/p$ being the parameter order and $\mathbf{\Gamma}^{-\frac{1}{p}}\left(\omega\right)$ is computed as

$$\mathbf{\Gamma}^{-\frac{1}{p}}\left(\omega\right) = \mathbf{U}\left(\omega\right)\mathbf{\Lambda}^{-\frac{1}{p}}\left(\omega\right)\mathbf{U}^T\left(\omega\right). \tag{18}$$

It follows then that the WNG of the parametric superdirective beamformer is

$$\mathcal{W}\left[\mathbf{h}_{\mathrm{P},p}\left(\omega\right)\right] = \frac{\left|\mathbf{h}_{\mathrm{P},p}^H\left(\omega\right)\mathbf{d}\left(\omega\right)\right|^2}{\mathbf{h}_{\mathrm{P},p}^H\left(\omega\right)\mathbf{h}_{\mathrm{P},p}\left(\omega\right)} = \frac{\left[\mathbf{d}^H\left(\omega\right)\mathbf{\Gamma}^{-\frac{1}{p}}\left(\omega\right)\mathbf{d}\left(\omega\right)\right]^2}{\mathbf{d}^H\left(\omega\right)\mathbf{\Gamma}^{-\frac{2}{p}}\left(\omega\right)\mathbf{d}\left(\omega\right)}, \tag{19}$$

with

$$\mathcal{W}\left[\mathbf{h}_{\mathrm{P},1}\left(\omega\right)\right] = \frac{\left[\mathbf{d}^H\left(\omega\right)\mathbf{\Gamma}^{-1}\left(\omega\right)\mathbf{d}\left(\omega\right)\right]^2}{\mathbf{d}^H\left(\omega\right)\mathbf{\Gamma}^{-2}\left(\omega\right)\mathbf{d}\left(\omega\right)} \leq M \tag{20}$$

and

$$\mathcal{W}\left[\mathbf{h}_{\mathrm{P},\infty}\left(\omega\right)\right] = M. \tag{21}$$
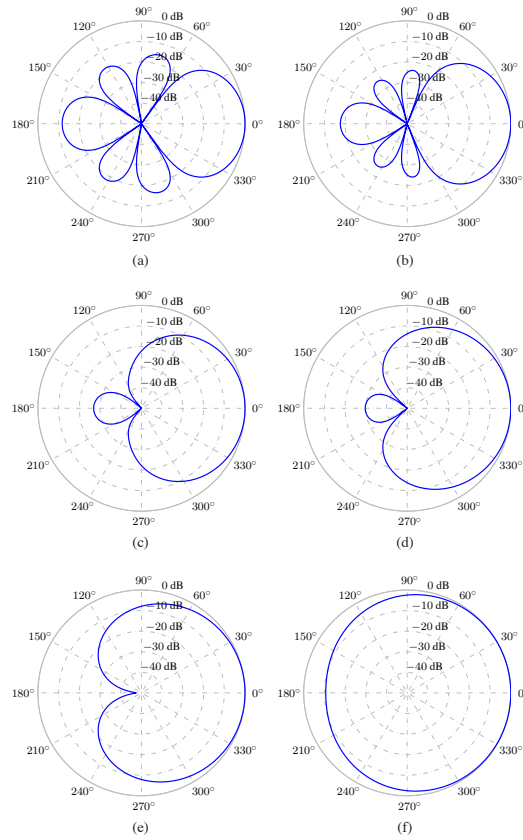
Figure 1: Beampatterns of the parametric superdirective beamformer with a uniform linear array, for six different values of p: (a) $p = 1$, (b) $p = 1.1$, (c) $p = 1.3$, (d) $p = 1.5$, (e) $p = 2$, and (f) $p = 10$. Conditions of simulation: $M = 4$, δ = 1.5 cm, and $f = 2000$ Hz.

The DF of this parametric superdirective beamformer is

$$\mathcal{D}\left[\mathbf{h}_{\mathrm{P},p}\left(\omega\right)\right] = \frac{\left|\mathbf{h}_{\mathrm{P},p}^{H}\left(\omega\right)\mathbf{d}\left(\omega\right)\right|^{2}}{\mathbf{h}_{\mathrm{P},p}^{H}\left(\omega\right)\boldsymbol{\Gamma}\left(\omega\right)\mathbf{h}_{\mathrm{P},p}\left(\omega\right)} = \frac{\left[\mathbf{d}^{H}\left(\omega\right)\boldsymbol{\Gamma}^{-\frac{1}{p}}\left(\omega\right)\mathbf{d}\left(\omega\right)\right]^{2}}{\mathbf{d}^{H}\left(\omega\right)\boldsymbol{\Gamma}^{1-\frac{2}{p}}\left(\omega\right)\mathbf{d}\left(\omega\right)}, \tag{22}$$

with

$$\mathcal{D}\left[\mathbf{h}_{\mathrm{P},1}\left(\omega\right)\right] = \mathbf{d}^{H}\left(\omega\right)\boldsymbol{\Gamma}^{-1}\left(\omega\right)\mathbf{d}\left(\omega\right) \leq M^{2} \tag{23}$$

and

$$\mathcal{D}\left[\mathbf{h}_{\mathrm{P},\infty}\left(\omega\right)\right] = \frac{M^{2}}{\mathbf{d}^{H}\left(\omega\right)\boldsymbol{\Gamma}\left(\omega\right)\mathbf{d}\left(\omega\right)} \geq 1. \tag{24}$$

For any given parameters $p_1 \geq p_2$, we should always have

$$\mathcal{W}\left[\mathbf{h}_{\mathrm{P},p_1}\left(\omega\right)\right] \geq \mathcal{W}\left[\mathbf{h}_{\mathrm{P},p_2}\left(\omega\right)\right] \tag{25}$$
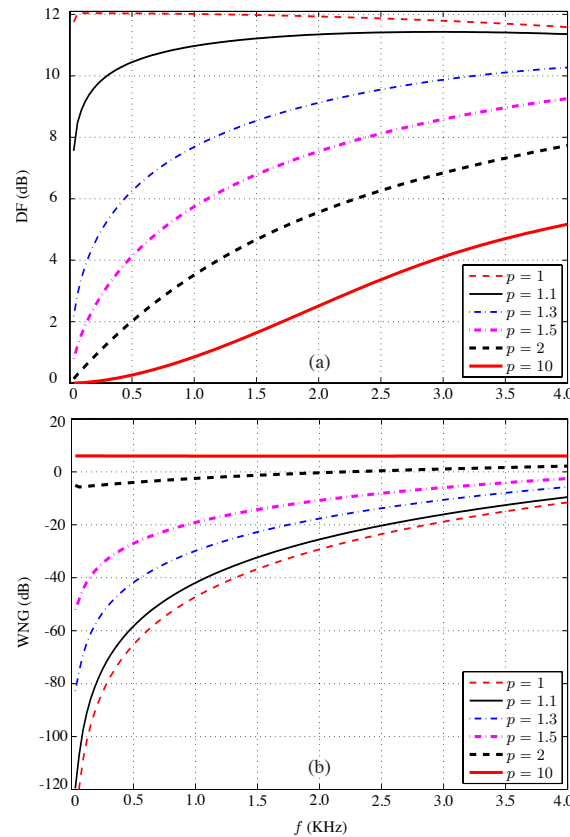
Figure 2: SNR gains of the parametric superdirective beamformer with a uniform linear array as a function of the frequency, f, for six different values of p: (a) DF and (b) WNG. Conditions of simulation: $M = 4$ and δ = 1.5 cm.

and

$$\mathcal{D}\left[\mathbf{h}_{\mathrm{P},p_1}\left(\omega\right)\right] \leq \mathcal{D}\left[\mathbf{h}_{\mathrm{P},p_2}\left(\omega\right)\right]. \tag{26}$$

Clearly, by playing with the value of the parameter p, we have three different cases:

- For $p = 1$, we obtain the conventional superdirective beamformer, i.e., $\mathbf{h}_{\mathrm{P},1}\left(\omega\right) = \mathbf{h}_{\mathrm{S}}\left(\omega\right)$, which achieves the maximum DF with a given number of sensors.

- For $p = \infty$, we get the DS beamformer, i.e., $\mathbf{h}_{\mathrm{P},\infty}\left(\omega\right) = \mathbf{h}_{\mathrm{DS}}\left(\omega\right)$, which achieves the maximum WNG.

- For $1 < p < \infty$, we obtain a beamformer whose DF decreases while WNG increases with p. So, by properly choosing the value of p, the parametric superdirective beamformer $\mathbf{h}_{\mathrm{P},p}\left(\omega\right)$ is able to control white noise amplification while having a reasonably good value of DF.
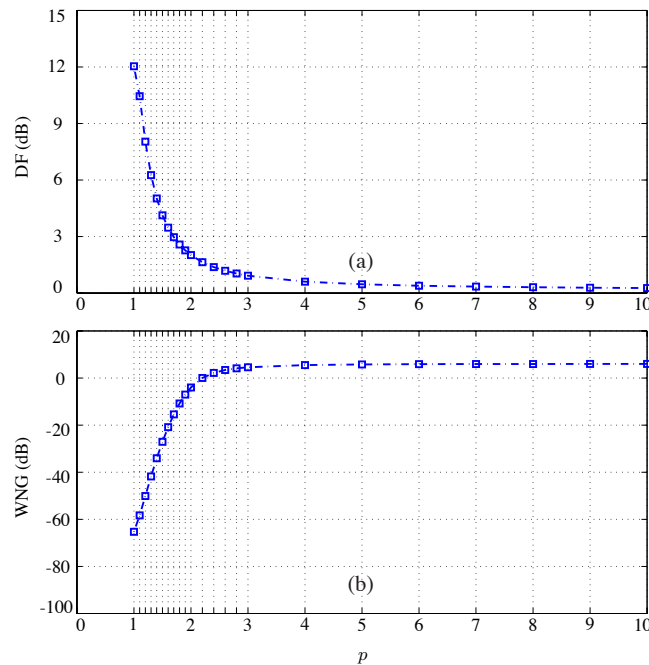
Figure 3: SNR gains of the parametric superdirective beamformer with a uniform linear array as a function of the parameter p: (a) DF and (b) WNG. Conditions of simulation: $M = 4$, $\delta =$ 1.5 cm, and $f = 1000$ Hz.

4 Simulations

In this section, we briefly study the performance of the parametric superdirective beamformer through simulations. We use a uniform linear array consisting of four closely spaced microphones, with $\delta = 1.5$ cm.

Figure 1 plots the beampatterns of the parametric superdirective beamformer with $p = 1$, 1.1, $1, 3$, 1.5, 2, 10 at frequency $f = 2000$ Hz. It is clearly seen that the patterns vary greatly with p. For $p = 1$, one can see that the beampattern is a superdirective one, which has a gain of one at the angle $\theta = 0°$ and three nulls in the range of $0°$ to $180°$. This superdirective patterns is the third-order hypercardioid as shown in [10] that the superdirective beamformer corresponds to the hypercardioid pattern of order $M - 1$ [10]. For a large value of p (e.g., $p = 10$), the beampattern resembles the DS beampattern.

Figure 2 plots both the WNG and the DF of the parametric superdirective beamformer (with $p = 1$, 1.1, $1, 3$, 1.5, 2, 10) as a function of the frequency, f. It is seen that the superdirective beamformer ($p = 1$) can achieve the maximum DF of approximately 12 dB. However, it is clearly

seen that it suffers from significant white noise amplification, particularly at low frequencies, which limits its use in practice. In comparison, the parametric superdirective beamformer can achieve a good compromise between large values of the DF and low values of the WNG. As seen, as the value of p increases from 1.1 to 10, the value of DF decreases while the value of WNG increases, which corroborates the theoretical analysis in Section 3.

Figure 3 plots the DF and the WNG as a function of p at $f = 1000$ Hz. It is seen that the DF decreases while the WNG increases significantly with p in the small range from 1 to 3. If the value of p is larger than 3, the DF and the WNG do not vary much. This indicates that we can choose a proper value of p within a small range to design the parametric superdirective beamformer even though theoretically this parameter can vary from 1 to positive infinity.

5   Conclusions

The superdirective beamformer, although maximizes the directivity factor, is found to suffer from white noise amplification, particularly significant at low frequencies. This means that the superdirective beamformer is very sensitive to sensors' noise and array imperfections. In this paper, we developed a parametric superdirective beamformer, which was obtained from the maximization of a parametric SNR gain. Simulation results demonstrated the good properties of this beamformer, which can compromise in a smooth way, thanks to and within a small range of the parameter p, between the DF and the WNG.

References

[1] H. Cox, R. Zeskind, and M. Owen, "Robust adaptive beamforming," *IEEE Trans. Acoust., Speech, Signal Process.*, vol. 35, pp. 1365–1376, Oct. 1987.

[2] J. Capon, "High-resolution frequency-wavenumber spectrum analysis," *Proc. IEEE*, vol. 57, pp. 1408–1418, Aug. 1969.

[3] A. Uzkov, "An approach to the problem of optimum directive antenna design," in *Comptes Rendus (Doklady) de l'Academie des Sciences de l'URSS*, pp. 35–38, 1946.

[4] C. Bouchard, D. I. Havelock, and M. Bouchard, "Beamforming with microphone arrays for directional sources," *J. Acoust. Soc. Am.*, vol. 125, no. 4, pp. 2098–2104, 2009.

[5] M. Crocco and A. Trucco, "The synthesis of robust broadband beamformers for equally-spaced linear arrays," *J. Acoust. Soc. Am.*, vol. 128, no. 2, pp. 691–701, 2010.

[6] J. Benesty, J. Chen, and Y. Huang, *Microphone Array Signal Processing*. Berlin, Germany: Springer-Verlag, 2008.

[7] H. Cox, R. M. Zeskind, and T. Kooij, "Practical supergain," *IEEE Trans. Acoust., Speech, Signal Process.*, vol. 34, no. 3, pp. 393–398, 1986.

[8] Y. Wang, Y. Yang, Y. Ma, and Z. He, "Robust high-order superdirectivity of circular sensor arrays," *J. Acoust. Soc. Am.*, vol. 136, no. 4, pp. 1712–1724, 2014.

[9] T. D. Abhayapala and A. Gupta, "Higher order differential-integral microphone arrays," *J. Acoust. Soc. Am.*, vol. 136, pp. 227–233, May 2010.

[10] J. Benesty and J. Chen, *Study and Design of Differential Microphone Arrays*. Berlin, Germany: Springer-Verlag, 2012.

[11] M. Crocco and A. Trucco, "Design of robust superdirective arrays with a tunable tradeoff between directivity and frequency-invariance," *IEEE Trans. Signal Process.*, vol. 59, no. 5, pp. 2169–2181, 2011.

[12] E. Mabande, A. Schad, and W. Kellermann, "Design of robust superdirective beamformers as a convex optimization problem," in *Proc. IEEE ICASSP*, pp. 77–80, 2009.

[13] J. Bitzer and K. U. Simmer, "Superdirective microphone arrays," in *Microphone Arrays*, pp. 19–38, Springer, 2001.

[14] S. Doclo and M. Moonen, "Superdirective beamforming robust against microphone mismatch," *IEEE Trans. Acoust., Speech, Signal Process.*, vol. 15, no. 2, pp. 617–631, 2007.

[15] G. W. Elko, "Superdirectional microphone arrays," in *Acoustic Signal Processing for Telecommunication*, pp. 181–237, Springer, 2000.

[16] R. Berkun, I. Cohen, and J. Benesty, "Combined beamformers for robust broadband regularized superdirective beamforming," *IEEE/ACM Trans. Audio, Speech, Lang. Process.*, vol. 23, no. 5, pp. 877–886, 2015.

[17] C. Li, J. Benesty, G. Huang, and J. Chen, "Subspace superdirective beamformers based on joint diagonalization," in *Proc. IEEE ICASSP*, 2016.

[18] G. H. Golub and C. F. V. Loan, *Matrix Computations. Third Edition*. Baltimore, Maryland: The Johns Hopkins University Press, 1996.

[19] M. Brandstein and D. Ward, *Microphone Arrays–Signal Processing Techniques and Applications*. Berlin, Germany: Springer-Verlag, 2001.