ON THE DESIGN OF FLEXIBLE KRONECKER PRODUCT BEAMFORMERS WITH LINEAR MICROPHONE ARRAYS

Wenxing Yang[1], *Gongping Huang*[1], *Jacob Benesty*[2], *Israel Cohen*[3], *and Jingdong Chen*[1]

[1]CIAIC, Northwestern Polytechnical University, Xi'an, Shaanxi 710072, China
[2]INRS-EMT, University of Quebec, Montreal, QC H5A 1K6, Canada
[3]Technion Israel Institute of Technology, Technion City, Haifa 32000, Israel

ABSTRACT

This paper proposes a method for the design of flexible Kronecker product beamformers based on the decomposition of the steering vector of a physical array as a Kronecker product of steering vectors of two smaller virtual arrays. With this decomposition, the global beamforming filter is designed by optimizing the two sub-beamformers in a cascaded manner, which can offer much flexibility to control the performance of beamforming or control the compromise between different, conflicted performance measures. In comparison with a recently developed method that restricts the number of microphones of the given physical array to a multiplication of two integers, each corresponding to the number of sensors of one virtual array, the approach in this work decomposes the physical array in such a way that the sensors in the two virtual arrays may share positions and the number of microphones of the physical array can be any positive integer. Simulations demonstrate the properties of the proposed approach.

***Index Terms*—**Microphone arrays, differential beamforming, Kronecker product, white noise gain, directivity factor.

1. INTRODUCTION

Microphone array beamforming has long been an important research topic due to its high potential in extracting the acoustic signals of interest and suppressing noise and interference in a wide range of applications [1–4]. Many interesting techniques have been developed, such as adaptive beamforming [5, 6], superdirective beamforming [7–11], differential beamforming [12–14], etc.

Generally, beamforming uses all microphones in the array in one step to form the optimal beamforming filter under some criteria [15–18]. In many applications, it is desirable to decompose the array into sub-arrays, each of which can be optimized individually [19–23]. Such decomposition usually offers more flexibility to control the performance of beamforming or control the compromise between different performance measures [20–23]. Recently, a differential Kronecker product beamformer is proposed [24], which decomposes the original uniform linear microphone array into two virtual uniform linear arrays. By optimizing the two sub-beamformers individually, the Kronecker product beamformer was demonstrated to be very flexible to design differential beamformers. One major limitation with this method is that it requires the number of microphones of the original array to be a multiplication of two integers, each corresponding to the number of sensors of one virtual array. So, the method cannot be applied to linear arrays with prime number of sensors. This paper extends the work in [24] and develops a more flexible solution to design differential beamformers with linear microphone arrays, which can be used in the general case where the number of microphones in the linear array can be any positive integer.

The rest of this paper is organized as follows. Section 2 describes the signal model, problem formulation of beamforming as well as some measures that are widely used to evaluate beamforming performance. An approach to beamforming with Kronecker product is presented in Section 3. Section 4 provides a design example to validate the developed method. Finally, some important conclusions are drawn in Section 5.

2. SIGNAL MODEL AND PERFORMANCE MEASURES

Consider a uniform linear microphone array consisting of M omnidirectional microphones and the distance between any two neighboring sensors is equal to δ. The direction of the source signal to the array is parameterized by the azimuth angle θ. In a far-field and anechoic acoustic environment, the steering vector corresponding to any direction θ is [25]

$$\mathbf{d}(\omega,\theta) = \begin{bmatrix} 1 & e^{-\jmath\varpi\cos\theta} & \cdots & e^{-\jmath(M-1)\varpi\cos\theta} \end{bmatrix}^T, \quad (1)$$

where the superscript ^{T} is the transpose operator, $\jmath$ is the imaginary unit with $\jmath^2 = -1$, $\varpi = \omega\delta/c$, $\omega = 2\pi f$ is the angular frequency, $f > 0$ is the temporal frequency, and c is the speed of sound in the air, which is typically assumed to be 340 m/s.

Beamforming consists of applying a complex weight vector:

$$\mathbf{h}(\omega) = \begin{bmatrix} H_1(\omega) & H_2(\omega) & \cdots & H_M(\omega) \end{bmatrix}^T \quad (2)$$

to the array observation vector to get an estimate of the source signal. Generally, in the desired look direction θ_s, the distortionless constraint is needed, i.e.,

$$\mathbf{h}^H(\omega)\,\mathbf{d}(\omega,\theta_s) = 1, \quad (3)$$

where the superscript ^{H} is the conjugate-transpose operator.

In order to evaluate the designed beampatterns, three performance measures, i.e., the beampattern, the white noise gain (WNG), and the directivity factor (DF), are typically used.

- Beampattern.

  The beampattern quantifies how the beamformer respond to a plane wave impinging on the array from the direction θ. It is

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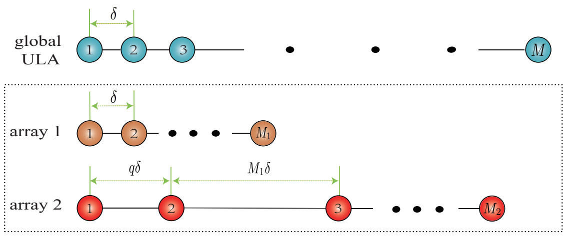

Fig. 1. Illustration of decomposition of a uniform linear array with M microphones and an interelement spacing of δ into two virtual arrays, where the first virtual uniform linear array consists of M_1 microphones with an interelement spacing of δ, and the second virtual non-uniform linear array consists of M_2 microphones.

defined as

$$\mathcal{B}\left[\mathbf{h}(\omega), \theta\right] = \mathbf{h}^H(\omega)\,\mathbf{d}(\omega, \theta) = \sum_{m=1}^{M} H_m^*(\omega)\, e^{-\jmath(m-1)\varpi\cos\theta}, \tag{4}$$

where the superscript * denotes complex conjugation.

- The white noise gain (WNG).

  The WNG evaluates the sensitivity of the beamformer to some array imperfections, such as sensor self noise. It is defined as [2]

$$\mathcal{W}\left[\mathbf{h}(\omega)\right] = \frac{\left|\mathbf{h}^H(\omega)\,\mathbf{d}(\omega, \theta_{\mathrm{s}})\right|^2}{\mathbf{h}^H(\omega)\,\mathbf{h}(\omega)}. \tag{5}$$

- The directivity factor (DF) and directivity index (DI).

  The DF, as its name indicates, describes how directive is the beamformer's spatial response. It also quantifies how the beamformer suppressing the spherically isotropic noise. Mathematically, it is defined as [2]

$$\mathcal{D}\left[\mathbf{h}(\omega)\right] = \frac{\left|\mathbf{h}^H(\omega)\,\mathbf{d}(\omega, \theta_{\mathrm{s}})\right|^2}{\mathbf{h}^H(\omega)\,\mathbf{\Gamma}_{\mathrm{d}}(\omega)\,\mathbf{h}(\omega)}, \tag{6}$$

  where the (i,j)th element of the matrix $\mathbf{\Gamma}_{\mathrm{d}}(\omega)$ is

$$\left[\mathbf{\Gamma}_{\mathrm{d}}(\omega)\right]_{ij} = \mathrm{sinc}\left(\frac{\omega\delta_{ij}}{c}\right), \tag{7}$$

  with $i, j = 1, 2, \ldots, M$, $\mathrm{sinc}(x) = \sin x / x$, and δ_{ij} being the distance between microphones i and j. The DI is simply the DF in the logarithmic scale [2], i.e.,

$$\mathrm{DI}\left[\mathbf{h}(\omega)\right] = 10\log_{10}\mathcal{D}\left[\mathbf{h}(\omega)\right]. \tag{8}$$

3. BEAMFORMING WITH KRONECKER PRODUCT

In [24], a differential Kronecker product beamformer was proposed, which is based on the Kronecker product decomposition of a physical linear array into two smaller virtual linear arrays. However, this method requires that the number of microphones in the global array be a multiplication of two integers. In this study, we attempt to consider a more general case, where the number of microphones in the global linear array, M, can be any positive integer.

Let us form the first virtual array as a uniform linear array with M_1 microphones and an interelement spacing of δ; the corresponding steering vector is

$$\mathbf{d}_1(\omega, \theta) = \begin{bmatrix} 1 & e^{-\jmath\varpi\cos\theta} & \cdots & e^{-\jmath(M_1-1)\varpi\cos\theta} \end{bmatrix}^T. \tag{9}$$

To figure out the geometry of the second virtual array, let us first decompose M as

$$M = pM_1 + q, \tag{10}$$

where $p \geq 1$ and $0 \leq q \leq M_1 - 1$ is a non-negative integer.

If $q = 0$, the second virtual array is constructed as a uniform linear array corresponding to a steering vector of

$$\mathbf{d}_2(\omega, \theta) = \begin{bmatrix} 1 & e^{-\jmath M_1\varpi\cos\theta} & \cdots & e^{-\jmath(p-1)M_1\varpi\cos\theta} \end{bmatrix}^T. \tag{11}$$

If $q \neq 0$, the second virtual array can be constructed as a nonuniform linear array corresponding to a steering vector of

$$\mathbf{d}_2(\omega, \theta) = \begin{bmatrix} 1 & e^{-\jmath q\varpi\cos\theta} & e^{-\jmath(q+M_1)\varpi\cos\theta} & \cdots & e^{-\jmath(q+(p-1)M_1)\varpi\cos\theta} \end{bmatrix}^T. \tag{12}$$

In both cases, the length of the vector $\mathbf{d}_2(\omega, \theta)$ is $\lceil M/M_1 \rceil$, with $\lceil x \rceil$ standing for the nearest integer greater than or equal to x. This implies that the second virtual array consists of $M_2 = \lceil M/M_1 \rceil$ microphones.

For the second virtual array, the distances of microphones to the reference (the first microphone) can be described as a vector as (see Fig. 1)

$$\begin{cases} \begin{bmatrix} 0 & M_1\delta & \cdots & (p-1)M_1\delta \end{bmatrix}^T, & \text{if } q = 0 \\ \begin{bmatrix} 0 & q\delta & (M_1+q)\delta & \cdots & ((p-1)M_1+q)\delta \end{bmatrix}^T, & \text{if } q \neq 0. \end{cases} \tag{13}$$

The Kronecker product of the steering vectors of these two virtual arrays is then

$$\overline{\mathbf{d}}(\omega, \theta) = \mathbf{d}_2(\omega, \theta) \otimes \mathbf{d}_1(\omega, \theta), \tag{14}$$

where $\otimes$ is the Kronecker product. We refer to the array that corresponds to the steering vector $\overline{\mathbf{d}}(\omega, \theta)$ as the produced array. It should be noted that we use $\overline{\mathbf{d}}(\omega, \theta)$ rather than $\mathbf{d}(\omega, \theta)$ to emphasize that the produced array is physically but may not be mathematically equivalent to the global array in (1).

Recall that the length of the vector $\mathbf{d}(\omega, \theta)$ corresponding to the global array is M. If $q = 0$, we have

$$M_1 M_2 = M_1\lceil M/M_1 \rceil = M_1\lceil pM_1/M_1 \rceil = M. \tag{15}$$

In this case, the produced array is equivalent to the global array, i.e., $\mathbf{d}(\omega, \theta) = \overline{\mathbf{d}}(\omega, \theta)$. If $q \neq 0$, we have

$$M_1 M_2 = M_1\lceil M/M_1 \rceil = M_1\lceil (pM_1 + q)/M_1 \rceil > M. \tag{16}$$

In this case, $\mathbf{d}(\omega, \theta) \neq \overline{\mathbf{d}}(\omega, \theta)$. This is because that some microphones may share the same position on the produced array. From this perspective, the produced array is still equivalent to the global array.

To see more clearly, a simple example is given as follows. Suppose that we want to design a Kronecker product beamformer with seven microphones, i.e. $M = 7$. If we set $M_1 = 3$, then $p = 2$,

$q=1$, and $M_2=3$. The steering vectors of the two virtual arrays are subsequently computed according to (9) and (12). For simplicity, we express the positions of microphones in the second and first virtual array as $\{0,1,4\}$ and $\{0,1,2\}$, respectively. Then, the positions of the microphones on the global array and the produced array can be expressed as

$$\begin{aligned} \text{global array}: &\ \{0,1,2,3,4,5,6\} \\ \text{produced array}: &\ \{0,1,4\}+\{0,1,2\} \\ &= \{0,1,2,1,2,3,4,5,6\}. \end{aligned} \tag{17}$$

We use "+" in (17) because the Kronecker product of two exponential functions means adding their exponential factors together. It is clearly seen that the 2nd and 4th, 3rd and 5th microphones on the produced array are overlapped. Therefore, the two arrays (vectors) are actually equivalent with each other.

We also write the beamforming filter corresponding to the produced array as a Kronecker product of two filters:

$$\overline{\mathbf{h}}(\omega)=\mathbf{h}_2(\omega)\otimes\mathbf{h}_1(\omega). \tag{18}$$

Consequently, with the Kronecker product formulation, we get a linear filter $\overline{\mathbf{h}}(\omega)$ of length M_1M_2. To make this beamforming filter applicable for the observation signal vector of length M on the global array, we need to either rearrange the observation signal vector or the beamforming filter.

Assume the observation signal vector received by the global array of length M is

$$\mathbf{y}(\omega) = \begin{bmatrix} Y_1(\omega) & Y_2(\omega) & \cdots & Y_M(\omega) \end{bmatrix}^T. \tag{19}$$

If $q=0$, the observation signal vector on the produced array is same as that of the global array, i.e., $\overline{\mathbf{y}}(\omega)=\mathbf{y}(\omega)$. If $q\neq 0$, the observation vector of length M_1M_2 on the produced array is constructed as

$$\overline{\mathbf{y}}(\omega) = \begin{bmatrix} \overline{Y}_1(\omega) & \overline{Y}_2(\omega) & \cdots & \overline{Y}_{M_1M_2}(\omega) \end{bmatrix}^T, \tag{20}$$

where

$$\overline{Y}_m(\omega)=\begin{cases} Y_m(\omega), & 1\le m\le M_1 \\ Y_{m-(M_1-q)}(\omega), & M_1<m\le M_1M_2. \end{cases} \tag{21}$$

The beamformer output is then

$$Z(\omega)=\overline{\mathbf{h}}^H(\omega)\overline{\mathbf{y}}(\omega). \tag{22}$$

Alternately, we can also keep $\mathbf{y}(\omega)$ unchanged but forming $\mathbf{h}(\omega)$ from $\overline{\mathbf{h}}(\omega)$analogously.

The beampattern corresponding to the produced array is [24]

$$\begin{aligned} \mathcal{B}\left[\overline{\mathbf{h}}(\omega),\theta\right] &= \overline{\mathbf{h}}^H(\omega)\overline{\mathbf{d}}(\omega,\theta) \qquad (23)\\ &= \left[\mathbf{h}_2(\omega)\otimes\mathbf{h}_1(\omega)\right]^H\left[\mathbf{d}_2(\omega,\theta)\otimes\mathbf{d}_1(\omega,\theta)\right] \\ &= \left[\mathbf{h}_2^H(\omega)\mathbf{d}_2(\omega,\theta)\right]\left[\mathbf{h}_1^H(\omega)\mathbf{d}_1(\omega,\theta)\right] \\ &= \mathcal{B}_2\left[\mathbf{h}_2(\omega),\theta\right]\times\mathcal{B}_1\left[\mathbf{h}_1(\omega),\theta\right], \end{aligned}$$

where $\mathcal{B}_1[\mathbf{h}_1(\omega),\theta]$ and $\mathcal{B}_2[\mathbf{h}_2(\omega),\theta]$ are the beampatterns corresponding to the first and second virtual array, respectively. As seen, the global beampattern, $\mathcal{B}\left[\overline{\mathbf{h}}(\omega),\theta\right]$, can be expressed as the product of the two beampatterns of the two smaller virtual arrays. This indicates that one can design the beampattern (or beamformer) separately as in a cascaded system instead of directly designing the differential-microphone-array (DMA) beamformer.

With the proposed method, it can be inferred that the WNG is the product of WNGs of two virtual arrays [24], i.e.,

$$\begin{aligned} \mathcal{W}\left[\overline{\mathbf{h}}(\omega)\right] &= \frac{\left|\overline{\mathbf{h}}^H(\omega)\overline{\mathbf{d}}(\omega,\theta_s)\right|^2}{\overline{\mathbf{h}}^H(\omega)\overline{\mathbf{h}}(\omega)} \\ &= \mathcal{W}_2\left[\mathbf{h}_2(\omega)\right]\times\mathcal{W}_1\left[\mathbf{h}_1(\omega)\right]. \end{aligned} \tag{24}$$

where $\mathcal{W}_2[\mathbf{h}_2(\omega)]$ and $\mathcal{W}_1[\mathbf{h}_1(\omega)]$ denote, respectively, the WNGs of two virtual arrays.

4. DESIGN EXAMPLE AND EVALUATION

Clearly, the Kronecker product beamforming method can be used to design any kind of beamformer. In this section, we show an example of the proposed Kronecker product beamformer on the design of some robust DMA beamformers.

We use the first virtual array to design the conventional DMAs. The Nth-order DMA directivity pattern with its main beam pointing to the direction of 0° is given by [1]

$$\mathcal{B}_N(\theta)=\sum_{n=0}^{N}a_{N,n}\cos^n\theta, \tag{25}$$

where $a_{N,n},\ n=0,1,\ldots,N$, are real coefficients. The values of the coefficients $a_{N,n}$, in (25) determine the different directivity patterns of the Nth-order DMA. In the direction of its main beam, i.e., $\theta=0^\circ$, the directivity pattern should be equal to 1, i.e., $\mathcal{B}(0^\circ)=1$. Therefore, the coefficients $a_{N,n}$ should satisfy

$$\sum_{n=0}^{N}a_{N,n}=1. \tag{26}$$

The objective of designing differential beamformer is to find an optimal filter under some criteria so that its beampattern is the same or as close as possible to the given target directivity pattern. It has been shown in [1] that a linear DMA can be designed based on the use of the nulls information in the range of $(0,\ 180^\circ]$ on the desired directivity pattern. Briefly, if we assume that the Nth-order directivity pattern has N nulls which satisfy $0^\circ<\theta_{N,1}<\cdots<\theta_{N,N}\le 180^\circ$, the problem of differential beamforming is transformed into one of solving the following linear system of equations [13]

$$\mathbf{D}(\omega)\mathbf{h}_{\mathrm{DMA}}(\omega)=\mathbf{i}_1, \tag{27}$$

where

$$\mathbf{D}(\omega)=\begin{bmatrix} \mathbf{d}_1^H(\omega,0^\circ) \\ \mathbf{d}_1^H(\omega,\theta_{N,1}) \\ \vdots \\ \mathbf{d}_1^H(\omega,\theta_{N,N}) \end{bmatrix} \tag{28}$$

is a constraint matrix of size $(N+1)\times(N+1)$, and $\mathbf{i}_1$ is a vector of length $(N+1)$, whose first element is 1 and all the other components are 0.

To design an Nth-order linear DMA, we assume that $M_1=N+1$. Then the solution to (27) is

$$\mathbf{h}_1(\omega)=\mathbf{h}_{\mathrm{DMA}}(\omega)=\mathbf{D}^{-1}(\omega)\mathbf{i}_1. \tag{29}$$

For the second virtual array, we choose the delay-and-sum (DS) beamformer:

$$\mathbf{h}_2(\omega)=\mathbf{h}_{\mathrm{DS}}(\omega)=\frac{1}{M_2}\mathbf{d}_2(\omega,\theta), \tag{30}$$

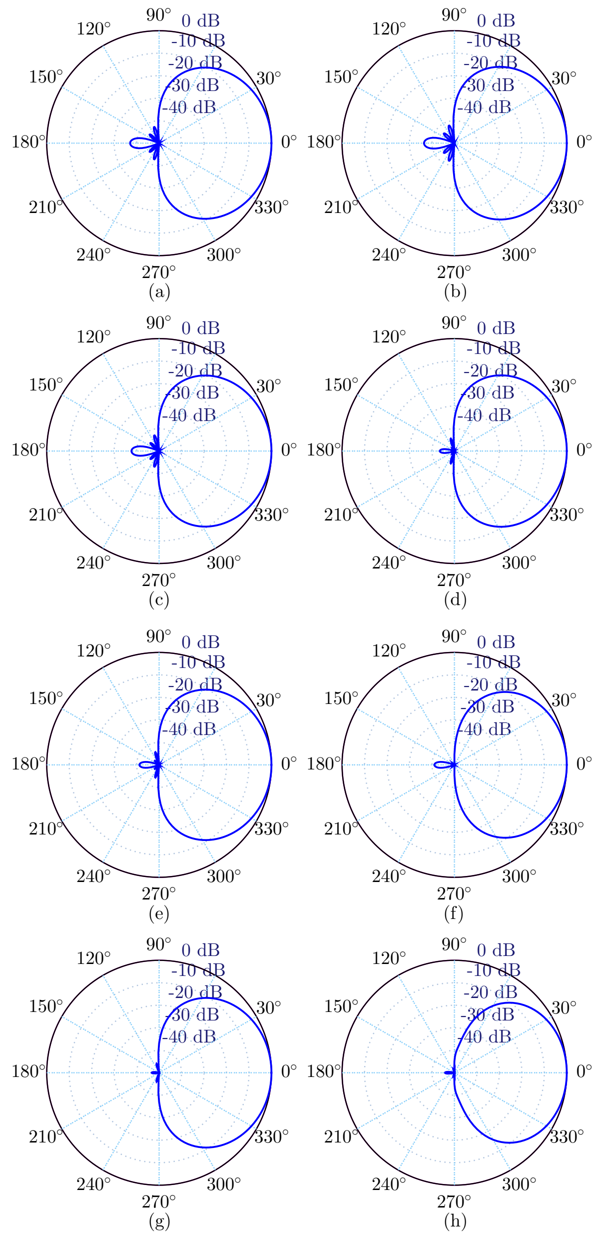

Fig. 2. Beampatterns of the proposed flexible Kronecker product beamformer on the design of a robust third-order supercardioid DMA: (a) $M = 4$, $f = 1$ kHz, (b) $M = 4$, $f = 3$ kHz, (c) $M = 6$, $f = 1$ kHz, (d) $M = 6$, $f = 3$ kHz, (e) $M = 10$, $f = 1$ kHz, (f) $M = 10$, $f = 3$ kHz, (g) $M = 13$, $f = 1$ kHz, (h) $M = 13$, $f = 3$ kHz. Conditions: $M_1 = 4$, $\delta = 1$cm.

which has the maximum WNG with the given number of microphones, i.e., $\mathcal{W}_2\left[\mathbf{h}_2(\omega)\right] = M_2$. Consequently, a robust DMA filter can be formed according to (18), i.e.,

$$\overline{\mathbf{h}}_{\rm RDMA}(\omega) = \mathbf{h}_2(\omega) \otimes \mathbf{h}_1(\omega) = \mathbf{h}_{\rm DS}(\omega) \otimes \mathbf{h}_{\rm DMA}(\omega), \quad (31)$$

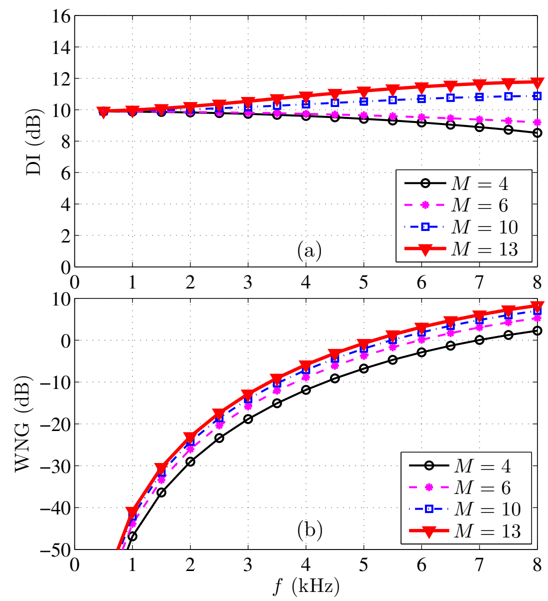

Fig. 3. DIs and WNGs of the designed robust third-order supercardioid DMA with the proposed method: (a) DIs and (b) WNGs. Conditions: $M_1 = 4$ and $\delta = 1$cm.

which forms an Nth-order DMA and maximizes the WGN at the same time. In implementation, the observation vector can be rearranged according to (20) and the beamforming process is implemented according to (22).

Figure 2 plots the beampatterns of the proposed robust DMA beamformer on the design of third-order supercardioid (with three nulls at 97°, 122°, and 153°) for $M \in \{4, 6, 10, 13\}$, at $f = 1$ kHz and 3 kHz, with $\delta = 1$ cm, $\theta_{\rm s} = 0^\circ$. Figure 3 plots the DIs and WNGs of the designed third-order supercardioid beamformer. It is clearly seen the WNG increases with the number of microphones. So, the more the number of microphones are used, the more robust is the designed beamformer with respect to array imperfections. Clearly, the proposed Kronecker product beamformer can be applied to the design of robust DMAs.

5. CONCLUSIONS

This paper proposed a flexible Kronecker product beamformer based on the Kronecker product decomposition of the given physical microphone array, where the number of microphones in the global array can be any positive integer. The physical array is decomposed into two virtual arrays, where the first one is a uniform linear array and the second may be a uniform or non-uniform linear array depending on the number of sensors in the global array as well as the number of sensors in the first virtual array. The resultant array is physically equivalent to the global array; but beamforming with the two arrays can be mathematically different due to the geometrical structure of the second virtual array. With this method, we can design the global beamforming system in a cascaded manner, which gives flexibility to control the beamforming performance. We also provided an example on the design of a robust DMA with the Kronecker product beamformer, which demonstrates the potential of the proposed approach for use in practice.

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