Hindawi Publishing Corporation
EURASIP Journal on Advances in Signal Processing
Volume 2007, Article ID 45194, 12 pages
doi:10.1155/2007/45194

Research Article

Recursive and Fast Recursive Capon Spectral Estimators

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Received 26 April 2006; Revised 13 November 2006; Accepted 11 December 2006

Recommended by Ulrich Heute

The Capon algorithm, which was originally proposed for wavenumber estimation in array signal processing, has become a powerful tool for spectral analysis. Over several decades, a significant amount of research attention has been devoted to the estimation of the Capon spectrum. Most of the developed algorithms thus far, however, rely on the direct computation of the inverse of the input correlation (or covariance) matrix, which can be computationally very expensive particularly when the dimension of the matrix is large. This paper deals with fast and efficient algorithms in computing the Capon spectrum. Inspired from the recursive idea established in adaptive signal processing theory, we first derive a recursive Capon algorithm. This new algorithm does not require an explicit matrix inversion, and hence it is more efficient to implement than the direct-inverse approach. We then develop a fast version of the recursive algorithm based on techniques used in fast recursive least-squares adaptive algorithms. This new fast algorithm can further reduce the complexity of the recursive Capon algorithm by an order of magnitude. Although our focus is on the Capon spectral estimation, the ideas shown in this paper can also be generalized and applied to other applications. To illustrate this, we will show how to apply the recursive idea to the estimation of the magnitude squared coherence function, which plays an important role for problems like time-delay estimation, signal-to-noise ratio estimation, and doubletalk detection in echo cancellation.

1. INTRODUCTION

Spectral estimation, which endeavors to determine the spectral content of a signal from a finite set of measurements, plays a major role in signal processing. It has a wide variety of applications in diversified fields such as radar, sonar, speech, communications, to name a few. Over the last century, a significant amount of research attention has been devoted to developing techniques for high performance spectral estimation. Some good reviews of such efforts can be found in [1–3]. Broadly, the developed techniques can be classified into two categories: nonparametric and parametric methods. The former is based on the concept of bandpass filtering. The latter assumes a model for the data, and the spectral estimation is then formulated into a problem of estimating the parameters in the assumed model. If the model fits the data well, the latter can yield more accurate spectral estimate than the former. In practice, however, the assumed model may not satisfy the data due to the lack of some a priori knowledge. In this case, the parametric method may suffer significant performance degradation and may even lead to biased estimation. Consequently, the nonparametric approach is still of great interest, and continues to be the focus of considerable studies, thanks to its robustness.

Among the numerous nonparametric techniques that were developed, the periodogram and Capon are perhaps the two most well-known approaches which have been widely used in various applications. The periodogram is essentially a discrete Fourier transform of the input data. It can be considered as a filter bank approach, which uses bandpass filters whose transfer functions are given by the discrete Fourier matrix.

In contrast to the periodogram, which uses signal-independent bandpass filters, the Capon method uses adaptive bandpass filters, where each filter is designed as selective as it can be according to the signal characteristics. This technique, also known as minimum variance distortionless response (MVDR), was originally proposed for frequency-wavenumber estimation in the late 1960s [4–6]. It has then been extensively studied in the literature and adopted for many applications such as spectral analysis [1–3, 7–12], beamforming [13–19], direction-of-arrival (DOA) estimation [20], speech analysis [21–24], and so forth. Since it uses data-dependent bandpass filters, the Capon approach can yield a higher spectral resolution than the periodogram, and therefore has been preferably used in various problems.

Many algorithms have been proposed in the literature to compute the Capon spectrum (see, e.g., [1–3, 25]). Most of these are block techniques: they first estimate the correlation (covariance) matrix of the signal when a block of measurements are available; the spectrum is then estimated based on inverting the correlation (covariance) matrix. Due to the high computational cost involved in the matrix inversion, such block approaches are computationally very expensive, particularly when the dimension of the matrix is large. This problem can become even more serious in applications like speech communication where a spectrum estimate has to be obtained every few milliseconds.

Recently, many efforts have been made to develop more efficient algorithms to compute the Capon spectrum [25–32]. One way of doing this is to borrow the well-established recursive idea from adaptive signal processing techniques and formulate the Capon approach into a recursive structure [33, 34], thereby, avoiding the direct computation of the inverse matrix. This is indeed the choice that we have taken in this paper, where a recursive Capon algorithm is derived, which obtains the inverse spectrum of the input signal on a sample-by-sample basis. This new algorithm does not require an explicit matrix inversion, and is, therefore, more efficient to implement than the direct-inverse approach. To further reduce the complexity and make the recursive Capon algorithm even more computationally efficient, a fast version of the recursive algorithm is developed, based on techniques used in fast recursive least-squares adaptive algorithms. We show that this new fast algorithm can reduce the complexity by an order of magnitude.

Although the main focus of this paper is on efficient computation of the Capon spectrum, the ideas shown here can also be generalized and applied to other applications. To illustrate this, we discuss how to extend the recursive Capon algorithm to the estimation of the magnitude squared coherence (MSC) function, which is very useful in practice [35, 36]. The most common way to estimate MSC is by Welch's method [37]. We will show how the Capon principle can be extended here for the derivation of a new estimation algorithm. Compared with Welch's method, this new algorithm can yield more accurate MSC estimates.

This paper is organized as follows. Section 2 revisits the Capon spectral estimator and shows how this approach is related to the periodogram. Even though this result is well-known and obvious, it is always interesting to present this link from another perspective. In Section 3, we present a way to recursively estimate the Capon inverse spectrum at time n from its estimate at time $n-1$. We will see how this recursion depends on the Kalman gain vector. In Section 4, we derive a fast algorithm, based on linear prediction techniques, to compute the recursive Capon inverse spectrum. In Section 5, we develop another interesting way to estimate the coherence function thanks to the Capon method. Section 6 presents some simulations and finally we draw our conclusions in Section 7.

2. CAPON SPECTRAL ESTIMATOR OVERVIEW

Before discussing the recursive and fast recursive Capon algorithms, let us briefly revisit the Capon spectral estimator. The Capon method for spectral estimation is based on a filterbank decomposition: the spectrum of a signal is estimated in each band by a simple filter design subject to some constraints [4, 6]. Let $x(n)$ be a zero-mean random process with a power spectral density (PSD) $S_{xx}(\omega)$, where $\omega(0 \leq \omega < 2\pi)$ is the angular frequency. Consider a complex-valued bandpass finite impulse response (FIR) filter

$$\mathbf{h}_k = \begin{bmatrix} h_{k,0} & h_{k,1} & \cdots & h_{k,L-1} \end{bmatrix}^T, \tag{1}$$

where L is the order of the filter and superscript ^{T} denotes the transpose of a vector or a matrix. If we pass $x(n)$ through this filter, the output of the filter at time n can be written as

$$y_k(n) = \mathbf{h}_k^H \mathbf{x}(n), \tag{2}$$

where ^{H} indicates conjugate transpose, and

$$\mathbf{x}(n) = \begin{bmatrix} x(n) & x(n-1) & \cdots & x(n-L+1) \end{bmatrix}^T \tag{3}$$

is a vector containing the L most recent samples of the observation signal $x(n)$. It follows immediately that

$$E\Big[\,|\,y_k(n)\,|^2\Big] = E\Big[\,|\,\mathbf{h}_k^H \mathbf{x}(n)\,|^2\Big] = \mathbf{h}_k^H \mathbf{R}_{xx}\,\mathbf{h}_k, \tag{4}$$

where $E[\cdot]$ denotes the mathematical expectation, and

$$\mathbf{R}_{xx} = E\big[\mathbf{x}(n)\mathbf{x}^H(n)\big] \tag{5}$$

is the covariance matrix of $x(n)$.

To find the Capon filter, we need to minimize (4) subject to the constraint that the filter has unity frequency response at $\omega = \omega_k = 2\pi k/K$, that is,

$$\mathbf{h}_k^{\mathrm{C}} = \arg\min_{\mathbf{h}_k} \mathbf{h}_k^H \mathbf{R}_{xx}\,\mathbf{h}_k \tag{6}$$

subject to

$$H_k(\omega_k) = \sum_{l=0}^{L-1} h_{k,l} \exp\left(-j\omega_k l\right) = \mathbf{f}_k^H \mathbf{h}_k = \mathbf{h}_k^H \mathbf{f}_k = 1, \tag{7}$$

where

$$\mathbf{f}_k = \begin{bmatrix} 1 & \exp\left(j\omega_k\right) & \cdots & \exp\left[j(L-1)\omega_k\right] \end{bmatrix}^T. \tag{8}$$

This optimization problem can also be formulated as the finding of a filter that minimizes the following cost function:

$$J_k = \mathbf{h}_k^H \mathbf{R}_{xx}\,\mathbf{h}_k + \mu\left[1 - \mathbf{h}_k^H \mathbf{f}_k\right], \tag{9}$$

where μ is a Lagrange multiplier. The solution to this is given by [4, 6]

$$\mathbf{h}_k^{\mathrm{C}} = \frac{\mathbf{R}_{xx}^{-1}\mathbf{f}_k}{\mathbf{f}_k^H\mathbf{R}_{xx}^{-1}\mathbf{f}_k}. \tag{10}$$

Substituting (10) into (4) yields

$$E\left[\,|\,y_k(n)\,|^2\right] = \frac{1}{\mathbf{f}_k^H\mathbf{R}_{xx}^{-1}\mathbf{f}_k}. \tag{11}$$

This is the power of $x(n)$ in the passband of the Capon filter centered on ω_k. The spectrum $S_{xx}(\omega)$ at frequency ω_k can then be determined as

$$S_{xx}(\omega_k) = \frac{c}{\mathbf{f}_k^H\mathbf{R}_{xx}^{-1}\mathbf{f}_k}, \tag{12}$$

where the factor c is added for properly scaling the Capon power estimator to obtain the spectral density. The scaling factor is typically determined based on the filter bandwidth. It can be seen from (10) that the Capon filter is data dependent, so the scaling factor may not necessarily be data and frequency independent. Many methods have been developed for determining the scaling factor c, and the most accurate one is the method provided in [38]. However, since our focus in this paper is on fast computation of Capon algorithm rather than the Capon spectral estimator itself, we take the simplest method given in [3] and set $c = K$, where K is the number of bandpass filters. For a good detailed discussion on this issue, the reader is invited to consult the papers by Lagunas et al. [38, 39].

From the previous analysis, we see that as long as $\mathbf{R}_{xx}^{-1}$ exists, we can compute the spectrum of $x(n)$ through (12). In practice, the covariance matrix $\mathbf{R}_{xx}$ has to be estimated. Suppose that we replace the expectation operation in (4) by the exponentially weighted sample average. We then have

$$\begin{aligned}\sum_{m=0}^{n}\lambda^{n-m}\,|\,y_k(m)\,|^2 &= \sum_{m=0}^{n}\lambda^{n-m}\,|\,\mathbf{h}_k^H\mathbf{x}(m)\,|^2 \\ &= \mathbf{h}_k^H\widehat{\mathbf{R}}_{xx}(n)\mathbf{h}_k,\end{aligned} \tag{13}$$

where λ $(0 < \lambda < 1)$ is a forgetting factor and

$$\widehat{\mathbf{R}}_{xx}(n) = \sum_{m=0}^{n}\lambda^{n-m}\mathbf{x}(m)\mathbf{x}^H(m) \tag{14}$$

is an estimate of the covariance matrix of $x(n)$. Following the same procedure from (4) to (12), we can obtain, respectively, an estimate of the Capon filter and spectrum at frequency ω_k and time n as

$$\widehat{\mathbf{h}}_k^{\mathrm{C}}(n) = \frac{\widehat{\mathbf{R}}_{xx}^{-1}(n)\mathbf{f}_k}{\mathbf{f}_k^H\widehat{\mathbf{R}}_{xx}^{-1}(n)\mathbf{f}_k}, \tag{15}$$

$$\widehat{S}_{xx}(\omega_k, n) = \frac{(1-\lambda)K}{\mathbf{f}_k^H\widehat{\mathbf{R}}_{xx}^{-1}(n)\mathbf{f}_k}. \tag{16}$$

We deduce from (15) and (16) that

$$K(1-\lambda)\widehat{\mathbf{R}}_{xx}(n)\widehat{\mathbf{h}}_k^{\mathrm{C}}(n) = \widehat{S}_{xx}(\omega_k, n)\mathbf{f}_k. \tag{17}$$

Taking into account all frequencies ω_k, $k = 0, 1, \ldots, K-1$, we can write (17) into the following general form:

$$K(1-\lambda)\widehat{\mathbf{R}}_{xx}(n)\widehat{\mathbf{H}}(n) = \mathbf{F}\widehat{\mathbf{S}}_{xx}(\omega_k, n), \tag{18}$$

where

$$\begin{gathered}\widehat{\mathbf{H}}(n) = \left[\widehat{\mathbf{h}}_0^{\mathrm{C}}(n)\ \ \widehat{\mathbf{h}}_1^{\mathrm{C}}(n)\ \ \cdots\ \ \widehat{\mathbf{h}}_{K-1}^{\mathrm{C}}(n)\right], \\ \mathbf{F} = \left[\mathbf{f}_0\ \ \mathbf{f}_1\ \ \cdots\ \ \mathbf{f}_{K-1}\right], \\ \widehat{\mathbf{S}}_{xx}(\omega_k, n) = \operatorname{diag}\{\widehat{S}_{xx}(\omega_0, n), \widehat{S}_{xx}(\omega_1, n), \ldots, \widehat{S}_{xx}(\omega_{K-1}, n)\}.\end{gathered} \tag{19}$$

For $K = L$, $\mathbf{F}$ is the Fourier matrix and $\mathbf{F}^H\mathbf{F} = K\mathbf{I}$ so $\mathbf{F}^{-1} = (1/K)\mathbf{F}^H$. We then obtain the following interesting decomposition:

$$\mathbf{F}^H\widehat{\mathbf{R}}_{xx}(n)\widehat{\mathbf{H}}(n) = \frac{1}{1-\lambda}\widehat{\mathbf{S}}_{xx}(\omega_k, n). \tag{20}$$

For a stationary signal, $\widehat{\mathbf{R}}_{xx}(n) \to 1/(1-\lambda)\mathbf{R}_{xx}$ when $n \to \infty$, where $\mathbf{R}_{xx}$, the true covariance matrix, is Toeplitz, and expression (20) becomes

$$\mathbf{F}^H\mathbf{R}_{xx}\,\widehat{\mathbf{H}} = \widehat{\mathbf{S}}_{xx}(\omega_k). \tag{21}$$

Now suppose $K \to \infty$. In this case, a Toeplitz matrix is asymptotically equivalent to a circulant matrix if its elements are absolutely summable [40], which is usually the case in most applications. Hence, we can decompose $\mathbf{R}_{xx}$ as

$$\mathbf{F}^{-1}\mathbf{R}_{xx}\,\mathbf{F} = \widehat{\mathbf{S}}_{xx}(\omega_k) = \mathbf{F}^H\mathbf{R}_{xx}\frac{\mathbf{F}}{K}. \tag{22}$$

Identifying (21) with (22), we see that $\widehat{\mathbf{H}} = (1/K)\mathbf{F}$. As a result, for a stationary signal, the Capon approach is asymptotically equivalent to the periodogram. The difference between the periodogram and Capon approaches can be viewed as the difference between the eigenvalue decompositions of circulant and Toeplitz matrices. While it is data independent for a circulant matrix, the unitary eigenvector matrix for a Toeplitz matrix is data dependent.

From (16), we find another interesting way to write (20),

$$\operatorname{diag}^{-1}\{\mathbf{F}^{-1}\widehat{\mathbf{R}}_{xx}^{-1}(n)\mathbf{F}\} = \frac{1}{1-\lambda}\widehat{\mathbf{S}}_{xx}(\omega_k, n), \tag{23}$$

where $\mathbf{F}^{-1}\widehat{\mathbf{R}}_{xx}^{-1}(n)\mathbf{F}$ is a diagonal matrix if and only if $\widehat{\mathbf{R}}_{xx}(n)$ is a circulant matrix.

3. A RECURSIVE COMPUTATION OF THE INVERSE SPECTRUM

The estimation of the Capon spectrum using (16) requires the computation of the inverse of the covariance matrix, which can be computationally very expensive. The aim of this section is to develop a recursion for the Capon algorithm so that the spectrum can be estimated more efficiently.

The covariance matrix of the signal $x(n)$ can be computed recursively,

$$\widehat{\mathbf{R}}_{xx}(n) = \lambda\widehat{\mathbf{R}}_{xx}(n-1) + \mathbf{x}(n)\mathbf{x}^H(n). \tag{24}$$

By using the matrix inversion lemma [33], $\widehat{\mathbf{R}}_{xx}^{-1}(n)$ can also be computed recursively:

$$\widehat{\mathbf{R}}_{xx}^{-1}(n) = \lambda^{-1}\widehat{\mathbf{R}}_{xx}^{-1}(n-1) - \lambda^{-2}\varphi(n)\mathbf{g}'(n)\mathbf{g}'^{H}(n), \quad (25)$$

where $\mathbf{g}'(n) = \widehat{\mathbf{R}}_{xx}^{-1}(n-1)\mathbf{x}(n)$ is the a priori Kalman gain vector and

$$\varphi(n) = \frac{\lambda}{\lambda + \mathbf{x}^{H}(n)\widehat{\mathbf{R}}_{xx}^{-1}(n-1)\mathbf{x}(n)}. \quad (26)$$

The a posteriori Kalman gain vector $\mathbf{g}(n) = \widehat{\mathbf{R}}_{xx}^{-1}(n)\mathbf{x}(n)$ is related to $\mathbf{g}'(n)$ by [41]

$$\mathbf{g}(n) = \lambda^{-1}\varphi(n)\mathbf{g}'(n). \quad (27)$$

Now, if we pre- and post-multiply both sides of (25) by $\mathbf{f}_k^H$ and $\mathbf{f}_k$, respectively, and with the help of (16), we get

$$\frac{(1-\lambda)K}{\widehat{S}_{xx}(\omega_k, n)} = \frac{\lambda^{-1}(1-\lambda)K}{\widehat{S}_{xx}(\omega_k, n-1)} - \lambda^{-2}\varphi(n)\left|\mathbf{f}_k^H\mathbf{g}'(n)\right|^2, \quad (28)$$

hence,

$$\widehat{S}_{xx}^{-1}(\omega_k, n) = \lambda^{-1}\widehat{S}_{xx}^{-1}(\omega_k, n-1) - \lambda^{-2}\varphi'(n)\left|\nu_{\mathrm{g},k}(n)\right|^2, \quad (29)$$

where

$$\begin{aligned}\varphi'(n) &= (1-\lambda)^{-1}K^{-1}\varphi(n),\\ \nu_{\mathrm{g},k}(n) &= \mathbf{f}_k^H\mathbf{g}'(n).\end{aligned} \quad (30)$$

Expression (29) shows how the inverse spectrum of the signal $x(n)$ at frequency ω_k and time n can be computed recursively from its value at time $n-1$, when a new data sample is available. This inverse spectrum depends on the a priori Kalman gain vector $\mathbf{g}'(n)$. Algorithm 1 summarizes this recursive algorithm where E_0 is the signal energy.

It can be seen from (29) that the estimation of the inverse Capon spectrum requires the computation of one inner product. The corresponding complexity is proportional to KL. For $K = L$, this complexity is proportional to L^2, which is quite high for practical applications. A natural question then arises: can we further reduce the number of operations of the recursive Capon algorithm or ideally make this number linear with respect to L? In the next section, we will discuss an efficient recursive Capon algorithm.

Initialization

$$\begin{aligned}&\mathbf{f}_k = \left[1 \ \exp(j\omega_k) \ \cdots \ \exp[j(L-1)\omega_k]\right]^T,\\ &\widehat{\mathbf{R}}_{xx}^{-1}(0) = E_0^{-1}\mathbf{I}, \text{ positive diagonal},\\ &\widehat{S}_{xx}^{-1}(\omega_k, 0) = \frac{L}{E_0(1-\lambda)K}, \quad \forall k.\end{aligned}$$

Kalman gain estimation

$$\begin{aligned}&\mathbf{g}'(n) = \widehat{\mathbf{R}}_{xx}^{-1}(n-1)\mathbf{x}(n),\\ &\varphi(n) = \frac{\lambda}{\lambda + \mathbf{x}^H(n)\mathbf{g}'(n)},\\ &\widehat{\mathbf{R}}_{xx}^{-1}(n) = \lambda^{-1}\widehat{\mathbf{R}}_{xx}^{-1}(n-1) - \lambda^{-2}\varphi(n)\mathbf{g}'(n)\mathbf{g}'^{H}(n),\\ &\varphi'(n) = \frac{\varphi(n)}{(1-\lambda)K}.\end{aligned}$$

Inv. spect. estimation

$$\begin{aligned}&k = 0, 1, \ldots, K-1,\\ &\nu_{\mathrm{g},k}(n) = \mathbf{f}_k^H\mathbf{g}'(n),\\ &\widehat{S}_{xx}^{-1}(\omega_k, n) = \lambda^{-1}\widehat{S}_{xx}^{-1}(\omega_k, n-1) - \lambda^{-2}\varphi'(n)\left|\nu_{\mathrm{g},k}(n)\right|^2.\end{aligned}$$

Algorithm 1: A recursive Capon inverse spectral estimator.

4. A FAST RECURSIVE ALGORITHM

In this section, we are going to show that the inner product $\nu_{\mathrm{g},k}(n) = \mathbf{f}_k^H\mathbf{g}'(n)$ can be computed recursively with a couple of multiplications only at each iteration, instead of L. As a result, the complexity of the entire algorithm will be reduced significantly since $\nu_{\mathrm{g},k}(n)$ has to be evaluated K times for every time sample n.

4.1. A fast algorithm based on linear prediction

It is well known that the a priori Kalman gain vector of order $L+1$ can be computed in two different ways [41]:

$$\begin{aligned}\mathbf{g}_{\mathrm{l}}'(n) &= \begin{bmatrix}\mathbf{g}'(n)\\ 0\end{bmatrix} + \frac{e_{\mathrm{b}}(n)}{E_{\mathrm{b}}(n-1)}\begin{bmatrix}-\mathbf{b}(n-1)\\ 1\end{bmatrix}\\ &= \begin{bmatrix}0\\ \mathbf{g}'(n-1)\end{bmatrix} + \frac{e_{\mathrm{a}}(n)}{E_{\mathrm{a}}(n-1)}\begin{bmatrix}1\\ -\mathbf{a}(n-1)\end{bmatrix},\end{aligned} \quad (31)$$

where $\mathbf{a}(n)$ and $\mathbf{b}(n)$ are, respectively, the forward and backward predictors of order L, $e_{\mathrm{a}}(n)$ and $e_{\mathrm{b}}(n)$ are the a priori forward and backward prediction error signals, and $E_{\mathrm{a}}(n)$ and $E_{\mathrm{b}}(n)$ are the forward and backward prediction error energies.

Consider the following vector of length $L+1$:

$$\begin{aligned}\mathbf{f}_{\mathrm{l},k} &= \left[1 \ \exp(j\omega_k) \ \cdots \ \exp(j\omega_k L)\right]^T\\ &= \left[\mathbf{f}_k^T \ \exp(j\omega_k L)\right]^T = \left[1 \ \exp(j\omega_k)\mathbf{f}_k^T\right]^T.\end{aligned} \quad (32)$$

If we pre-multiply both sides of (31) by $\mathbf{f}_{\mathrm{l},k}^H$, we obtain the recursion

$$\begin{aligned}\nu_{\mathrm{g},k}(n) &= \exp(-j\omega_k)\nu_{\mathrm{g},k}(n-1)\\ &\quad - \frac{e_{\mathrm{b}}(n)}{E_{\mathrm{b}}(n-1)}\left[\exp(-j\omega_k L) - \nu_{\mathrm{b},k}(n-1)\right]\\ &\quad + \frac{e_{\mathrm{a}}(n)}{E_{\mathrm{a}}(n-1)}\left[1 - \exp(-j\omega_k)\nu_{\mathrm{a},k}(n-1)\right],\end{aligned} \quad (33)$$

where

$$\begin{aligned}\nu_{\mathrm{a},k}(n-1) &= \mathbf{f}_k^H\mathbf{a}(n-1),\\ \nu_{\mathrm{b},k}(n-1) &= \mathbf{f}_k^H\mathbf{b}(n-1).\end{aligned} \quad (34)$$

In order for (33) to be efficient, (34) need to be computed recursively. This can be easily done thanks to the update equations of the forward and backward predictors

$$\mathbf{a}(n) = \mathbf{a}(n-1) + \frac{e_{\mathrm{a}}^{*}(n)}{\alpha(n-1)}\mathbf{g}'(n-1),$$
$$\mathbf{b}(n) = \mathbf{b}(n-1) + \frac{e_{\mathrm{b}}^{*}(n)}{\alpha(n)}\mathbf{g}'(n), \tag{35}$$

where the superscript $*$ is the complex conjugate operator and $\alpha(n) = \lambda/\varphi(n)$. Now, if we pre-multiply both sides of (35) by $\mathbf{f}_k^H$, we deduce the two recursions

$$\nu_{\mathrm{a},k}(n) = \nu_{\mathrm{a},k}(n-1) + \frac{e_{\mathrm{a}}^{*}(n)}{\alpha(n-1)}\nu_{\mathrm{g},k}(n-1),$$
$$\nu_{\mathrm{b},k}(n) = \nu_{\mathrm{b},k}(n-1) + \frac{e_{\mathrm{b}}^{*}(n)}{\alpha(n)}\nu_{\mathrm{g},k}(n). \tag{36}$$

The two previous expressions should be used in (33). Therefore, the inner product $\nu_{\mathrm{g},k}(n)$ can be estimated with roughly six complex multiplications, instead of L. Algorithm 2 shows an example of a fast recursive Capon inverse spectral estimator.

Initialization

$$\mathbf{g}'(0) = \mathbf{a}(0) = \mathbf{b}(0) = \mathbf{0},$$
$$\alpha(0) = \lambda,$$
$$E_{\mathrm{a}}(0) = E_0, \text{ positive constant},$$
$$E_{\mathrm{b}}(0) = E_0\lambda^{-L},$$
$$\nu_{\mathrm{a},k}(0) = \nu_{\mathrm{b},k}(0) = \nu_{\mathrm{g},k}(0) = 0, \quad \forall k,$$
$$S_{xx}^{-1}(\omega_k, 0) = \frac{L}{E_0(1-\lambda)K}, \quad \forall k.$$

Prediction

$$e_{\mathrm{a}}(n) = x(n) - \mathbf{a}^H(n-1)\mathbf{x}(n-1),$$
$$\alpha_1(n) = \alpha(n-1) + \frac{|e_{\mathrm{a}}(n)|^2}{E_{\mathrm{a}}(n-1)},$$
$$\begin{bmatrix}\mathbf{t}(n)\\ m(n)\end{bmatrix} = \begin{bmatrix}0\\ \mathbf{g}'(n-1)\end{bmatrix} + \begin{bmatrix}1\\ -\mathbf{a}(n-1)\end{bmatrix}\frac{e_{\mathrm{a}}(n)}{E_{\mathrm{a}}(n-1)},$$
$$E_{\mathrm{a}}(n) = \lambda\left[E_{\mathrm{a}}(n-1) + \frac{|e_{\mathrm{a}}(n)|^2}{\alpha(n-1)}\right],$$
$$\mathbf{a}(n) = \mathbf{a}(n-1) + \mathbf{g}'(n-1)\frac{e_{\mathrm{a}}^{*}(n)}{\alpha(n-1)},$$
$$e_{\mathrm{b}}(n) = x(n-L) - \mathbf{b}^H(n-1)\mathbf{x}(n),$$
$$\mathbf{g}'(n) = \mathbf{t}(n) + \mathbf{b}(n-1)m(n),$$
$$\alpha(n) = \alpha_1(n) - e_{\mathrm{b}}^{*}(n)m(n),$$
$$E_{\mathrm{b}}(n) = \lambda\left[E_{\mathrm{b}}(n-1) + \frac{|e_{\mathrm{b}}(n)|^2}{\alpha(n)}\right],$$
$$\mathbf{b}(n) = \mathbf{b}(n-1) + \mathbf{g}'(n)\frac{e_{\mathrm{b}}^{*}(n)}{\alpha(n)},$$
$$\varphi'(n) = \frac{\varphi(n)}{(1-\lambda)K}.$$

Inv. spect. estimation

$$k = 0, 1, \dots, K-1,$$
$$\nu_{\mathrm{a},k}(n-1) = \nu_{\mathrm{a},k}(n-2) + \frac{e_{\mathrm{a}}^{*}(n-1)}{\alpha(n-2)}\nu_{\mathrm{g},k}(n-2),$$
$$\nu_{\mathrm{b},k}(n-1) = \nu_{\mathrm{b},k}(n-2) + \frac{e_{\mathrm{b}}^{*}(n-1)}{\alpha(n-1)}\nu_{\mathrm{g},k}(n-1),$$
$$\nu_{\mathrm{g},k}(n) = \exp(-j\omega_k)\nu_{\mathrm{g},k}(n-1) - \frac{e_{\mathrm{b}}(n)}{E_{\mathrm{b}}(n-1)}\left[\exp(-j\omega_k L) - \nu_{\mathrm{b},k}(n-1)\right] + \frac{e_{\mathrm{a}}(n)}{E_{\mathrm{a}}(n-1)}\left[1 - \exp(-j\omega_k)\nu_{\mathrm{a},k}(n-1)\right],$$
$$\hat{S}_{xx}^{-1}(\omega_k, n) = \lambda^{-1}\hat{S}_{xx}^{-1}(\omega_k, n-1) - \lambda^{-2}\varphi'(n)|\nu_{\mathrm{g},k}(n)|^2.$$

Algorithm 2: A fast recursive Capon inverse spectral estimator.

4.2. Complexity analysis

Now let us compare the computational complexity of the direct-inverse, the recursive, and the fast recursive Capon algorithms. Here the computational complexity is evaluated in terms of the number of real-valued multiplications/divisions required for the implementation of each algorithm. The number of additions/subtractions are neglected because they are much quicker to compute in most generic hardware platforms. We assume that complex-valued multiplications are transformed into real-valued multiplications. The multiplication between a real and complex numbers requires 2 real-valued multiplications. The multiplication between two complex numbers needs 4 real-valued multiplications. The division between a complex number and a real number requires 2 real-valued multiplications.

Suppose that the observation signal is real-valued and a spectral estimate has to be made every N samples. The direct-inverse approach achieves the spectral estimate in two steps. It first computes the signal correlation matrix according to (14). This step requires $N(L^2 + L)$ multiplications. It then estimates the Capon spectrum using (16). If we assume that the inverse of the correlation matrix is computed through LU decomposition, which requires $L^3 - L$ multiplications [42], it is trivial to show that the second step involves $L^3 - L + K(2L^2 + 4L + 1)$ multiplications. We deduce the grand total, for estimating the inverse spectrum by direct-inverse method, of $L^3 + (N + 2K)L^2 + (N + 4K - 1)L + K$ multiplications.

For the recursive algorithm, the inverse spectrum depends on the a priori Kalman gain vector $\mathbf{g}'(n)$, which can be computed at each iteration, using linear prediction techniques, as shown in Algorithm 2. This step involves $16L + 15$ multiplications. Equation (29), which requires the calculation of one inner product, involves $(2L+4)K$ multiplications for estimating the inverse spectra for all frequencies ω_k, $k = 0, 1, \dots, K-1$. The total cost for N samples is, therefore, $N(2LK + 16L + 4K + 15)$ multiplications.

The fast recursive algorithm also requires computation of the a priori Kalman gain vector $\mathbf{g}'(n)$, which involves $16L+15$ multiplications. But this technique can estimate the inverse spectrum with only $20K$ multiplications after knowing the Kalman gain vector. The total complexity for N samples is, therefore, $N(16L + 15 + 20K)$.

If we assume $N = L = K$, the computational complexities for the direct-inverse, recursive, and fast recursive Capon

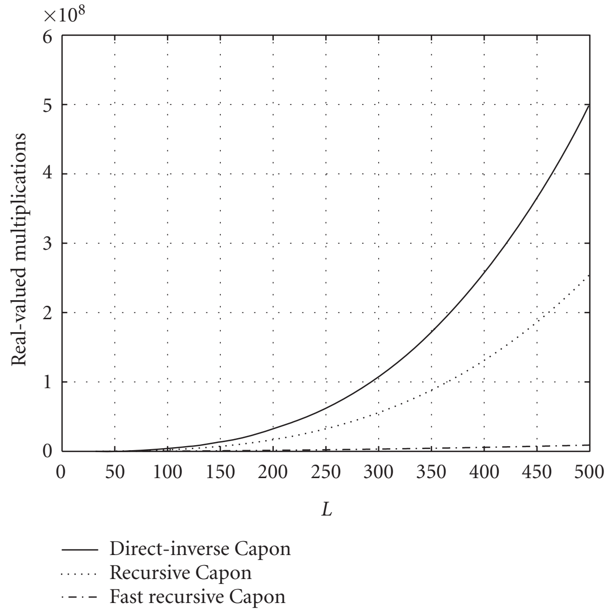

Figure 1: Comparison of computational complexity among the direct-inverse, the recursive, and the fast recursive Capon algorithms for different L's in the condition where $N = L = K$.

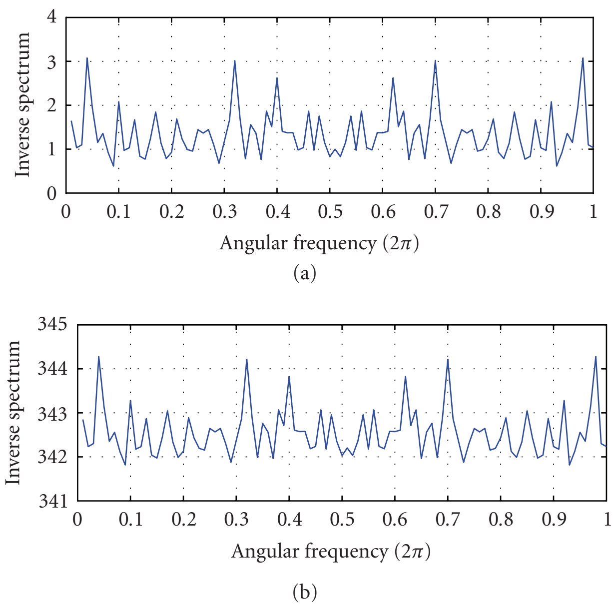

Figure 2: Inverse spectrum of a white Gaussian signal with $\sigma_x^2 = 1$, $L = K = 100$, $\lambda = 1 - 1/(5L)$, and $N = 1000$. (a) Recursive algorithm. (b) Fast recursive algorithm.

algorithms are, respectively, $4L^3 + 5L^2$, $2L^3 + 20L^2 + 15L$, and $36L^2 + 15L$ multiplications. Figure 1 plots these results as a function of L. As can be seen clearly, The recursive algorithm is computationally less expensive than the direct-inverse approach and the fast recursive algorithm is much more efficient than both the recursive and the direct inverse approach.

4.3. Discussion on the bias removal

Having discussed the complexity issue, we now examine the error-propagation effect, another important problem that needs much attention when any fast algorithm is developed.

From Algorithm 2, we see that the proposed fast algorithm requires two initializations. One is for the inverse spectrum $[\hat{S}_{xx}^{-1}(\omega_k, 0)]$. The other is for the prediction error energies $[E_a(0)$ and $E_b(0)$, which are involved in the efficient computation of the Kalman gain vector], which in turn is used to estimate the inverse spectrum. Both initializations depend on the energy (E_0) of the signal $x(n)$. Because of this interlink process and the fact that the two initializations are not perfectly synchronized, one interesting phenomenon appears during the update: a bias is introduced in the inverse spectrum estimation, which grows with the time index n. As an example, we illustrate this phenomenon with a zero-mean white Gaussian signal $x(n)$ with a variance of $\sigma_x^2 = 1$, $L = K = 100$, $\lambda = 1 - 1/(5L)$, and for $N = 1000$. Figure 2 shows the estimates of the inverse spectrum with the recursive [Figure 2(a)] and fast recursive [Figure 2(b)] algorithms. We can notice that the two inverse spectra are identical but the y-axis scale is different. This difference is due to the bias. Therefore, inverting the estimate obtained with the fast recursive algorithm will give a wrong result for the spectrum estimation of the signal $x(n)$.

From the recursive expression of the inverse spectrum given in (29), we have

$$\begin{aligned}\hat{S}_{xx}^{-1}(\omega_k, n) &= \lambda^{-1}\hat{S}_{xx}^{-1}(\omega_k, n-1) - \lambda^{-2}\varphi'(n)\left|v_{g,k}(n)\right|^2 \\ &= \lambda^{-2}\hat{S}_{xx}^{-1}(\omega_k, n-2) - \lambda^{-3}\varphi'(n-1)\left|v_{g,k}(n-1)\right|^2 \\ &\quad - \lambda^{-2}\varphi'(n)\left|v_{g,k}(n)\right|^2.\end{aligned} \tag{37}$$

Continuing this recursion, we deduce that

$$\begin{aligned}&\hat{S}_{xx}^{-1}(\omega_k, n) \\ &\quad = \frac{L\lambda^{-n}}{(1-\lambda)KE_0} - \lambda^{-2}\sum_{m=0}^{n-1}\lambda^{-m}\varphi'(n-m)\left|v_{g,k}(n-m)\right|^2.\end{aligned} \tag{38}$$

The first term of the right-hand side of the previous equation depends on the initialization E_0^{-1}. The second term depends on E_0 as well; but with the fast algorithm, this dependency diminishes as n gets larger. Therefore, the overall bias in the inverse spectrum is from the first term. Now suppose that we initialize the inverse spectrum with $E_1 = E_0 - \delta$ (where δ is a small positive number and $\delta \ll E_1$). Replacing E_0 in (38) with E_1 and using the approximation $1/(1-\delta/E_0) \approx 1+\delta/E_0$, we get the corresponding spectrum estimate,

$$
\begin{aligned}
\widetilde{S}_{xx}^{-1}(\omega_k, n) &= \frac{L\lambda^{-n}}{(1-\lambda)KE_0} + \frac{\delta L\lambda^{-n}}{(1-\lambda)KE_0^2} \\
&\quad - \lambda^{-2}\sum_{m=0}^{n-1}\lambda^{-m}\varphi'(n-m)\left|v_{\mathrm{g},k}(n-m)\right|^2 \\
&= \frac{\delta L\lambda^{-n}}{(1-\lambda)KE_0^2} + \widehat{S}_{xx}^{-1}(\omega_k, n) \\
&= \Delta_{\mathrm{b}}(n) + \widehat{S}_{xx}^{-1}(\omega_k, n).
\end{aligned}
\tag{39}
$$

The term $\Delta_{\mathrm{b}}(n)$ represents the bias which grows exponentially with n. Expression (39) shows very clearly that even a very small mismatch between E_0 and E_1 (or equivalently a very small δ) will introduce an avoidable bias. Obviously, a technique is needed to remove it.

According to (39), we have

$$
\begin{aligned}
\frac{1}{K}\sum_{k=0}^{K-1}\widehat{S}_{xx}(\omega_k, n) &= \frac{1}{K}\sum_{k=0}^{K-1}\left[\widetilde{S}_{xx}^{-1}(\omega_k, n) - \Delta_{\mathrm{b}}(n)\right]^{-1} \\
&= (1-\lambda)\sum_{m=0}^{n}\lambda^{n-m}\left|x(m)\right|^2 = \sigma_x^2(n), \\
\Delta_{\mathrm{b}}(n) &< \min_k \widetilde{S}_{xx}^{-1}(\omega_k, n).
\end{aligned}
\tag{40}
$$

By taking into account the information from the two previous expressions, the following simple iterative algorithm:

$$
\begin{aligned}
e(n,i) &= \frac{K}{L} - \frac{1}{K\sigma_x^2(n)}\sum_{k=0}^{K-1}\left[\widetilde{S}_{xx}^{-1}(\omega_k, n) - \Delta_{\mathrm{b}}(n, i-1)\right]^{-1}, \\
\Delta_{\mathrm{b}}(n,i) &= \Delta_{\mathrm{b}}(n,i-1) + \frac{0.1}{\sigma_x^2(n)}e(n,i),
\end{aligned}
\tag{41}
$$

with $\Delta_{\mathrm{b}}(n,0) = \min_k \widetilde{S}_{xx}^{-1}(\omega_k, n) - 1/\sigma_x^2(n)$, finds the bias with less than 100 iterations. Let us take again the example given at the beginning of this section. Figure 3 shows the spectrum estimated with the recursive algorithm [Figure 3(a)] and the spectrum estimated with the fast recursive algorithm [Figure 3(b)] after the bias was removed with the proposed iterative method. From this example, it is clear that the obtained solution is quite satisfactory.

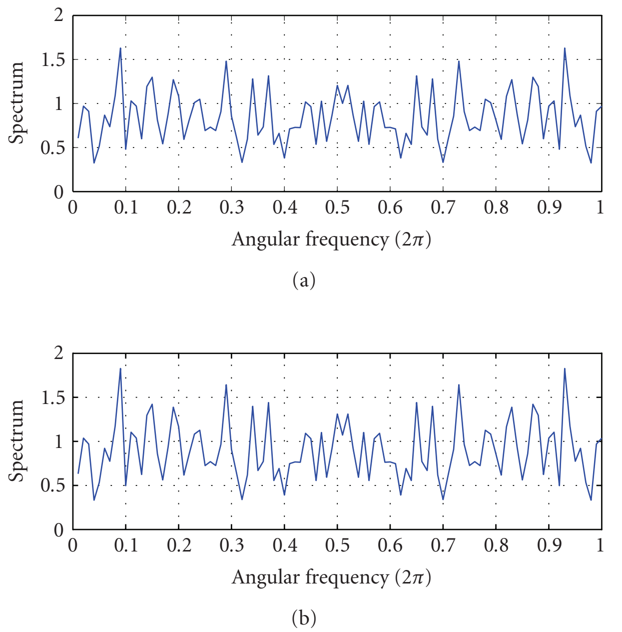

Figure 3: Spectrum of a white Gaussian signal with $\sigma_x^2 = 1$, $L = K = 100$, $\lambda = 1 - 1/(5L)$, and $N = 1000$. (a) Recursive algorithm. (b) Fast recursive algorithm after bias removal.

5. MAGNITUDE SQUARED COHERENCE FUNCTION ESTIMATION WITH THE CAPON METHOD

The aforementioned recursive and fast recursive ideas can also be generalized and applied to other applications. For example, we extend in this section the recursive idea to the estimation of the magnitude squared coherence function, which plays an important role for problems like time-delay estimation, signal-to-noise ratio estimation, and doubletalk detection in echo cancellation.

We assume here that we have two zero-mean random signals $x_1(n)$ and $x_2(n)$ with respective spectra $S_{x_1x_1}(\omega, n)$ and $S_{x_2x_2}(\omega, n)$. As explained in Section 2, we can design two Capon bandpass filters,

$$
\widehat{\mathbf{h}}_{p,k}^{\mathrm{C}}(n) = \frac{\widehat{\mathbf{R}}_{x_px_p}^{-1}(n)\mathbf{f}_k}{\mathbf{f}_k^H\widehat{\mathbf{R}}_{x_px_p}^{-1}(n)\mathbf{f}_k}, \quad p = 1, 2, \tag{42}
$$

to find the spectra of $x_1(n)$ and $x_2(n)$ at frequency ω_k and at time n:

$$
\widehat{S}_{x_px_p}(\omega_k, n) = \frac{(1-\lambda)K}{\mathbf{f}_k^H\widehat{\mathbf{R}}_{x_px_p}^{-1}(n)\mathbf{f}_k}, \quad p = 1, 2, \tag{43}
$$

where

$$
\widehat{\mathbf{R}}_{x_px_p}(n) = \sum_{m=0}^{n}\lambda^{n-m}\mathbf{x}_p(m)\mathbf{x}_p^H(m) \tag{44}
$$

is an estimate of the covariance matrix of the signal $x_p(n)$ and

$$
\mathbf{x}_p(m) = \left[x_p(m) \quad x_p(m-1) \quad \cdots \quad x_p(m-L+1)\right]^T, \quad p = 1, 2. \tag{45}
$$

Let $y_{1,k}(n)$ and $y_{2,k}(n)$ be the respective outputs of the filters $\widehat{\mathbf{h}}_{1,k}^{\mathrm{C}}(n)$ and $\widehat{\mathbf{h}}_{2,k}^{\mathrm{C}}(n)$. The cross-spectrum between $x_1(n)$ and $x_2(n)$ at frequency ω_k is [38, 43, 44]

$$
\widehat{S}_{x_1x_2}(\omega_k, n) = K(1-\lambda)\sum_{m=0}^{n}\lambda^{n-m}y_{1,k}(m)y_{2,k}^*(m). \tag{46}
$$

Similarly,

$$
\begin{aligned}
\widehat{S}_{x_2x_1}(\omega_k, n) &= K(1-\lambda)\sum_{m=0}^{n}\lambda^{n-m}y_{2,k}(m)y_{1,k}^*(m) \\
&= \widehat{S}_{x_1x_2}^*(\omega_k, n).
\end{aligned}
\tag{47}
$$

Now if we develop (46), we get

$$\widehat{S}_{x_1x_2}(\omega_k, n) = K(1-\lambda)\mathbf{h}_{1,k}^H(n)\widehat{\mathbf{R}}_{x_1x_2}(n)\mathbf{h}_{2,k}(n), \tag{48}$$

where

$$\widehat{\mathbf{R}}_{x_1x_2}(n) = \sum_{m=0}^{n} \lambda^{n-m}\mathbf{x}_1(m)\mathbf{x}_2^H(m) \tag{49}$$

is an estimate of the cross-correlation matrix between $x_1(n)$ and $x_2(n)$. Replacing (42) in (48), we obtain

$$\widehat{S}_{x_1x_2}(\omega_k, n) = K(1-\lambda)\frac{\mathbf{f}_k^H\widehat{\mathbf{R}}_{x_1x_1}^{-1}(n)\widehat{\mathbf{R}}_{x_1x_2}(n)\widehat{\mathbf{R}}_{x_2x_2}^{-1}(n)\mathbf{f}_k}{[\mathbf{f}_k^H\widehat{\mathbf{R}}_{x_1x_1}^{-1}(n)\mathbf{f}_k][\mathbf{f}_k^H\widehat{\mathbf{R}}_{x_2x_2}^{-1}(n)\mathbf{f}_k]}, \tag{50}$$

which does not depend on $\widehat{\mathbf{h}}_{p,k}^{\mathrm{C}}(n)$; hence,

$$\left|\widehat{S}_{x_1x_2}(\omega_k, n)\right|^2 = K^2(1-\lambda)^2\frac{\left|\mathbf{f}_k^H\widehat{\mathbf{R}}_{x_1x_1}^{-1}(n)\widehat{\mathbf{R}}_{x_1x_2}(n)\widehat{\mathbf{R}}_{x_2x_2}^{-1}(n)\mathbf{f}_k\right|^2}{[\mathbf{f}_k^H\widehat{\mathbf{R}}_{x_1x_1}^{-1}(n)\mathbf{f}_k]^2[\mathbf{f}_k^H\widehat{\mathbf{R}}_{x_2x_2}^{-1}(n)\mathbf{f}_k]^2}. \tag{51}$$

The magnitude squared coherence (MSC) function between two signals $x_1(n)$ and $x_2(n)$ is estimated as

$$\widehat{\gamma}_{x_1x_2}^2(\omega_k, n) = \frac{\left|\widehat{S}_{x_1x_2}(\omega_k, n)\right|^2}{\widehat{S}_{x_1x_1}(\omega_k, n)\widehat{S}_{x_2x_2}(\omega_k, n)}. \tag{52}$$

Plugging (43) and (51) into (52), we obtain the MSC estimate

$$\widehat{\gamma}_{x_1x_2}^2(\omega_k, n) = \frac{\left|\mathbf{f}_k^H\widehat{\mathbf{R}}_{x_1x_1}^{-1}(n)\widehat{\mathbf{R}}_{x_1x_2}(n)\widehat{\mathbf{R}}_{x_2x_2}^{-1}(n)\mathbf{f}_k\right|^2}{[\mathbf{f}_k^H\widehat{\mathbf{R}}_{x_1x_1}^{-1}(n)\mathbf{f}_k][\mathbf{f}_k^H\widehat{\mathbf{R}}_{x_2x_2}^{-1}(n)\mathbf{f}_k]}. \tag{53}$$

There are two extreme cases:

(i) $x_1(n) = x_2(n) = x(n)$ (the two signals are perfectly correlated). In this case, it is easy to verify from (53) that $\widehat{\gamma}_{x_1x_2}^2(\omega_k, n) = 1$, for all k, n.

(ii) $\widehat{\mathbf{R}}_{x_1x_2}(n) = \mathbf{0}$ (the two signals are completely uncorrelated), we have $\widehat{\gamma}_{x_1x_2}^2(\omega_k, n) = 0$, for all k, n.

Expression (53) requires an arithmetic complexity proportional to L^2K for all K frequencies and at each time instant n. This complexity can be reduced if (53) is computed recursively. For the two terms in the denominator, recursions can be derived in a similar way to the one developed in Section 3. A recursive expression for the term in the numerator is a little bit less obvious to obtain and several steps are necessary. We first define the normalized cross-correlation matrix as

$$\widehat{\mathbf{R}}_{\mathrm{n},x_1x_2}(n) = \widehat{\mathbf{R}}_{x_1x_1}^{-1}(n)\widehat{\mathbf{R}}_{x_1x_2}(n)\widehat{\mathbf{R}}_{x_2x_2}^{-1}(n). \tag{54}$$

Replacing the recursions,

$$\widehat{\mathbf{R}}_{x_ix_p}(n) = \lambda\widehat{\mathbf{R}}_{x_ix_p}(n-1) + \mathbf{x}_i(n)\mathbf{x}_p^H(n), \quad i, p = 1, 2, \tag{55}$$

in (54) and after some simple manipulations, we find

$$\begin{aligned}\widehat{\mathbf{R}}_{\mathrm{n},x_1x_2}(n) &= \lambda^{-1}[\mathbf{I} - \lambda^{-1}\varphi_1(n)\mathbf{g}_1'(n)\mathbf{x}_1^H(n)] \\ &\quad\times\widehat{\mathbf{R}}_{\mathrm{n},x_1x_2}(n-1)[\mathbf{I} - \lambda^{-1}\varphi_2(n)\mathbf{x}_2(n)\mathbf{g}_2'^H(n)] \\ &\quad+ \lambda^{-2}\varphi_1(n)\varphi_2(n)\mathbf{g}_1'(n)\mathbf{g}_2'^H(n) \\ &= \lambda^{-1}\widehat{\mathbf{R}}_{\mathrm{n},x_1x_2}(n-1) - \mathbf{g}_1'(n)\mathbf{v}_{\mathrm{n},x_1}^H(n) \\ &\quad- \mathbf{v}_{\mathrm{n},x_2}(n)\mathbf{g}_2'^H(n) + \beta_{\mathrm{n}}'(n)\mathbf{g}_1'(n)\mathbf{g}_2'^H(n),\end{aligned} \tag{56}$$

where

$$\begin{aligned}\mathbf{g}_p'(n) &= \widehat{\mathbf{R}}_{x_px_p}^{-1}(n-1)\mathbf{x}_p(n), \quad p = 1, 2, \\ \varphi_p(n) &= \frac{\lambda}{\lambda + \mathbf{x}_p^H(n)\mathbf{g}_p'(n)}, \quad p = 1, 2, \\ \mathbf{v}_{\mathrm{n},x_1}(n) &= \lambda^{-2}\varphi_1(n)\widehat{\mathbf{R}}_{\mathrm{n},x_1x_2}^H(n-1)\mathbf{x}_1(n), \\ \mathbf{v}_{\mathrm{n},x_2}(n) &= \lambda^{-2}\varphi_2(n)\widehat{\mathbf{R}}_{\mathrm{n},x_1x_2}(n-1)\mathbf{x}_2(n), \\ \beta_{\mathrm{n}}(n) &= \mathbf{x}_1^H(n)\widehat{\mathbf{R}}_{\mathrm{n},x_1x_2}(n-1)\mathbf{x}_2(n), \\ \beta_{\mathrm{n}}'(n) &= \lambda^{-2}[\lambda^{-1}\beta_{\mathrm{n}}(n) + 1]\varphi_1(n)\varphi_2(n).\end{aligned} \tag{57}$$

Define the variable

$$\begin{aligned}\zeta(\omega_k, n) &= \mathbf{f}_k^H\widehat{\mathbf{R}}_{x_1x_1}^{-1}(n)\widehat{\mathbf{R}}_{x_1x_2}(n)\widehat{\mathbf{R}}_{x_2x_2}^{-1}(n)\mathbf{f}_k \\ &= \mathbf{f}_k^H\widehat{\mathbf{R}}_{\mathrm{n},x_1x_2}(n)\mathbf{f}_k.\end{aligned} \tag{58}$$

If we pre- and post-multiply both sides of (56) by $\mathbf{f}_k^H$ and $\mathbf{f}_k$, respectively, we get

$$\begin{aligned}\zeta(\omega_k, n) &= \lambda^{-1}\zeta(\omega_k, n-1) - [\mathbf{f}_k^H\mathbf{g}_1'(n)][\mathbf{v}_{\mathrm{n},x_1}^H(n)\mathbf{f}_k] \\ &\quad- [\mathbf{f}_k^H\mathbf{v}_{\mathrm{n},x_2}(n)][\mathbf{g}_2'^H(n)\mathbf{f}_k] \\ &\quad+ \beta_{\mathrm{n}}'(n)[\mathbf{f}_k^H\mathbf{g}_1'(n)][\mathbf{g}_2'^H(n)\mathbf{f}_k].\end{aligned} \tag{59}$$

Compared to (58), where products of matrices and vectors are required, the previous expression manipulates inner products only.

Algorithm 3 summarizes this recursive algorithm for the estimation of the MSC function with the Capon approach. The overall complexity, for all K frequencies, is proportional to $L^2 + KL$ at each time sample n. For $L = K$, we see that we decreased this complexity by a factor of L.

6. EXPERIMENTS

Throughout the text, we have developed a recursive and a fast version of the recursive algorithm for estimating the Capon spectrum. In both cases, the spectral components are initialized equally across all the frequency bands. For signals with white spectrum, it has already been demonstrated that the fast algorithm (may require bias removal) can compute the Capon spectrum accurately and efficiently. To further verify the fast algorithm and the effect of the initialization condition on the spectral estimation, we give another example of applying the fast recursive algorithm to estimate the PSD of an ARMA process. The ARMA process is given by [2]

$$\begin{aligned}x(n) &= -2.760x(n-1) + 3.809x(n-2) - 2.654x(n-3) \\ &\quad+ 0.924x(n-4) + u(n) - 0.900u(n-1) \\ &\quad+ 0.810u(n-2),\end{aligned} \tag{60}$$

Initialization

$$\mathbf{f}_k = \left[1 \ \exp\left(j\omega_k\right) \ \cdots \ \exp\left[j(L-1)\omega_k\right]\right]^T,$$
$$\widehat{\mathbf{R}}_{x_p x_p}^{-1}(0) = E_p^{-1}\mathbf{I}, \quad p = 1, 2,$$
$$\widehat{\mathbf{R}}_{x_1 x_2}(0) = \widehat{\mathbf{R}}_{\mathrm{n}, x_1 x_2}(0) = \mathbf{0},$$
$$\varsigma_p(\omega_k, 0) = E_p^{-1}L, \quad p = 1, 2, \ \forall k,$$
$$\zeta(\omega_k, 0) = 0, \quad \forall k.$$

Correlation estimation

$$\mathbf{g}_p'(n) = \widehat{\mathbf{R}}_{x_p x_p}^{-1}(n-1)\mathbf{x}_p(n), \quad p = 1, 2,$$
$$\varphi_p(n) = \frac{\lambda}{\lambda + \mathbf{x}_p^H(n)\mathbf{g}_p'(n)}, \quad p = 1, 2,$$
$$\mathbf{v}_{\mathrm{n}, x_1}(n) = \lambda^{-2}\varphi_1(n)\widehat{\mathbf{R}}_{\mathrm{n}, x_1 x_2}^H(n-1)\mathbf{x}_1(n),$$
$$\mathbf{v}_{\mathrm{n}, x_2}(n) = \lambda^{-2}\varphi_2(n)\widehat{\mathbf{R}}_{\mathrm{n}, x_1 x_2}(n-1)\mathbf{x}_2(n),$$
$$\beta_{\mathrm{n}}(n) = \mathbf{x}_1^H(n)\widehat{\mathbf{R}}_{\mathrm{n}, x_1 x_2}(n-1)\mathbf{x}_2(n),$$
$$\beta_{\mathrm{n}}'(n) = \lambda^{-2}\left[\lambda^{-1}\beta_{\mathrm{n}}(n) + 1\right]\varphi_1(n)\varphi_2(n),$$
$$\begin{aligned}\widehat{\mathbf{R}}_{x_p x_p}^{-1}(n) = {} & \lambda^{-1}\widehat{\mathbf{R}}_{x_p x_p}^{-1}(n-1) \\ & -\lambda^{-2}\varphi(n)\mathbf{g}_p'(n)\mathbf{g}_p'^H(n), \quad p = 1, 2,\end{aligned}$$
$$\begin{aligned}\widehat{\mathbf{R}}_{\mathrm{n}, x_1 x_2}(n) = {} & \lambda^{-1}\widehat{\mathbf{R}}_{\mathrm{n}, x_1 x_2}(n-1) - \mathbf{g}_1'(n)\mathbf{v}_{\mathrm{n}, x_1}^H(n) \\ & -\mathbf{v}_{\mathrm{n}, x_2}(n)\mathbf{g}_2'^H(n) + \beta_{\mathrm{n}}'(n)\mathbf{g}_1'(n)\mathbf{g}_2'^H(n).\end{aligned}$$

MSC estimation

$$k = 0, 1, \ldots, K-1,$$
$$\begin{aligned}\varsigma_p(\omega_k, n) = {} & \lambda^{-1}\varsigma_p(\omega_k, n-1) \\ & -\lambda^{-2}\varphi_p(n)\left|\mathbf{f}_k^H\mathbf{g}_p'(n)\right|^2, \quad p = 1, 2,\end{aligned}$$
$$\begin{aligned}\zeta(\omega_k, n) = {} & \lambda^{-1}\zeta(\omega_k, n-1) - \left[\mathbf{f}_k^H\mathbf{g}_1'(n)\right]\left[\mathbf{v}_{\mathrm{n}, x_1}^H(n)\mathbf{f}_k\right] \\ & -\left[\mathbf{f}_k^H\mathbf{v}_{\mathrm{n}, x_2}(n)\right]\left[\mathbf{g}_2'^H(n)\mathbf{f}_k\right] \\ & +\beta_{\mathrm{n}}'(n)\left[\mathbf{f}_k^H\mathbf{g}_1'(n)\right]\left[\mathbf{g}_2'^H(n)\mathbf{f}_k\right],\end{aligned}$$
$$\widehat{\gamma}_{x_1 x_2}^2(\omega_k, n) = \frac{\left|\zeta(\omega_k, n)\right|^2}{\varsigma_1(\omega_k, n)\varsigma_2(\omega_k, n)}.$$

Algorithm 3: An MSC estimation based on the Capon method.

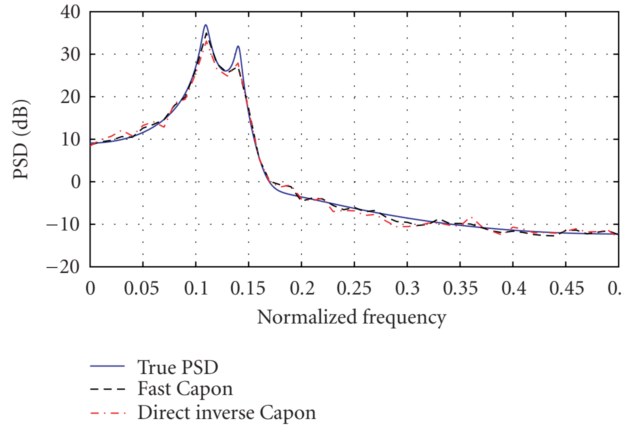

Figure 4: True and estimated power spectral densities for an ARMA process. Conditions of simulations: $L = K = 100$, $\lambda = 1 - 1/(5L)$, and $N = 2000$.

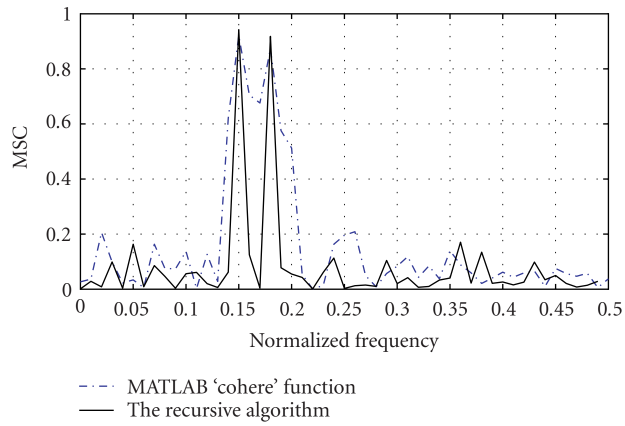

Figure 5: Estimation of the magnitude squared coherence function using both the Welch and the proposed approaches. Conditions of simulations: $L = K = 100$, $N = 1024$, and $\lambda = 1 - 1/(5L)$.

where $u(n)$ is a real white Gaussian noise with variance $\sigma_u^2 = 1$.

The true PSD of this ARMA process and the corresponding PSD estimates obtained with the Capon method using both the direct inverse and fast algorithms (initialization of the fast algorithm is based on the signal energy E_0. In our case, the estimated E_0 is 130.0. Bias removal is used) are plotted in Figure 4. The simulation conditions are $L = K = 100$, $\lambda = 1 - 1/(5L)$, and $N = 2000$. It is clearly seen that the fast algorithm, same as the direct inverse method, can estimate the PSD correctly.

In the second experiment, we would like to compare the MSC function estimated with our approach and with the MATLAB "cohere" function, which uses the Welch averaged periodogram method [37]. Let us consider the illustrative example of two signals $x_1(n)$ and $x_2(n)$ [44]:

$$\begin{aligned}x_1(n) &= w_1(n) + \cos\left(2\pi\Omega_{11}n\right) + \cos\left(2\pi\Omega_{12}n\right), \\ x_2(n) &= w_2(n) + \cos\left[2\pi\left(\Omega_{21}n + \phi_1\right)\right] + \cos\left[2\pi\left(\Omega_{22} + \phi_2\right)\right],\end{aligned} \tag{61}$$

where $w_1(n)$ and $w_2(n)$ are two independent white Gaussian random processes with zero mean and unit variance. The phases ϕ_1 and ϕ_2 in the signal $x_2(n)$ are random. We consider two cases. In the first one, we chose $\Omega_{11} = \Omega_{21} = 0.15$ and $\Omega_{12} = \Omega_{22} = 0.18$. In this situation, the theoretical coherence between these two signals should be equal to 1 at the two frequencies 0.15 and 0.18 and 0 at the others. For both the Welch and the developed new algorithms we took $N = 1024$ samples. The window length is $L = K = 100$, and $\lambda = 1 - 1/(5L)$. Figure 5 gives the MSC estimated using the MATLAB and new methods, respectively. It is clearly seen from the results that the estimation of the coherence function with the new algorithm is much closer to its theoretical values.

For the second case, we choose $\Omega_{11} = \Omega_{21} = 0.15$, $\Omega_{12} = 0.18$, and $\Omega_{22} = 0.185$. So the theoretical coherence between $x_1(n)$ and $x_2(n)$ should be equal to 1 at $\Omega_{11} = \Omega_{21} = 0.15$ and 0 at the other frequencies. In this experiment, we assume that the observation data sequence is short, say, $N = 256$ samples. For both the Welch and the new algorithms, we set $L = 50$, $K = 100$, and $\lambda = 1 - 1/(5L)$. The result is shown in Figure 6.

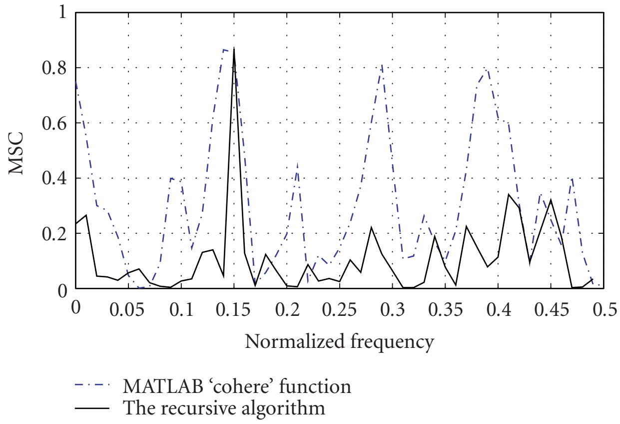

Figure 6: Estimation of the magnitude squared coherence function using both the Welch and the proposed approaches. Conditions of simulations: $L = 50$, $K = 100$, $N = 256$, and $\lambda = 1 - 1/(5L)$.

Again we see that the estimation of the coherence function with the new algorithm is more accurate.

7. CONCLUSIONS

The Capon algorithm has become a powerful tool for spectral analysis and many other applications. Over several decades, a significant amount of research attention has been devoted to the estimation of the Capon spectrum. Most of the developed algorithms thus far, however, rely on the direct computation of the inverse of the input correlation (or covariance) matrix. If the length of the Capon filter is L, the complexity of the direct-inverse approach is on the order of L^3. Such a high computational load makes the Capon algorithm difficult to implement in applications like speech communication where a spectral estimate has to be obtained every few milliseconds. In this paper, we derived a recursive Capon algorithm. This algorithm does not require an explicit matrix inversion, and hence is more efficient to implement than the direct-inverse approach. However, its complexity is still on the order of L^3. In order to further reduce the complexity and make the recursive Capon algorithm more computationally efficient, a fast version of the recursive algorithm was developed, based on the techniques used in the fast recursive least-squares adaptive algorithms. We showed that this new fast algorithm can reduce the complexity by an order of magnitude. Although we focused on the Capon spectral estimation, the ideas shown in this paper can also be generalized and applied to other applications. As an example, we extended the recursive idea to the estimation of the magnitude squared coherence (MSC) function, resulting in a new MSC estimator, which can achieve a higher estimation accuracy than the widely used Welch method.

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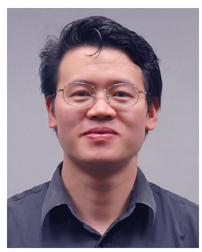

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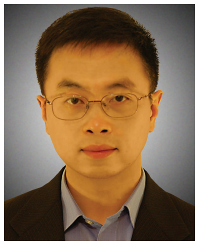

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